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ABSTRACT

This is part one of a three-part School Mathematics Study Group (SMSG) textbook. This part contains the first five chapters including: (1) Analytic Geometry, (2) Coordinates and the Line, (3) Vectors and Their Applications, (4) Proof by Analytic Methods, and (5) Graphs and Their Equations. (MK)

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Student Text
Part 1
(revised edition)

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ANALYTIC GEOMETRY

Student Text

Part 1

(revised edition)

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PREFACE

This text aims at restoring what is, in a sense, a "lost" subject. There is a widespread practice of including analytic geometry material in the calculus program; but when this is accomplished, Analytic Geometry, as a course of study, disappears and what remains of it is the part immediately useful to a study of calculus. You will find a much more varied selection of topics in this book than you would see in a calculus course.

In a book devoted to the interplay between algebra and geometry you would expect to be called upon to exhibit considerable dexterity in algebraic manipulations as well as to recall previous experiences with geometric figures and theorems. You will not be disappointed. It is also assumed that you know the elementary notions of trigonometry.

A deliberate effort was made to tie this text to previous SMSG texts; so, you will find the usual language of sets, ordered pairs, number properties, etc., with which you have had some acquaintance. This flavor is perhaps what distinguishes this book from others in the field. For example, the treatment of coordinate systems in Chapter 2 depends upon the postulates of SMSG Geometry.

Here is one word of advice. The early chapters are fundamental to everything which follows. Study them until they seem to be old friends; do not hesitate to return to them later for a fresh look. Another thing you might watch. The related ideas of vectors, direction numbers, and parameters are used extensively to simplify and unify the various topics. Look for this feature.

The theorems and figures are numbered serially within each chapter; e.g., Theorem 8-3 is the third theorem of Chapter 8, Figure 5-2 is the second figure to appear in Chapter 5. If an equation is to be referred to, it is assigned a counting number, which is then displayed in the left margin. The counting begins at one for each section. Definitions are not numbered but may be found by referring to the Index.

The writers hope they have recreated the beauty of Analytic Geometry in a new SMSG setting, and they further hope that you will enjoy and profit by the adventure you are about to undertake. Bon Voyage.

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Chapter 1

ANALYTIC GEOMETRY

1-1. What Is Analytic Geometry?

Geometry has been studied systematically for over two thousand years. Euclid's Elements, which was written about 300 B.C., is perhaps the most influential mathematics textbook ever published. There are undoubtedly many traces of it to be found in the text you used in your high school course.

Until the 17th century, geometry was studied by what are known as synthetic methods. The postulates dealt with such geometric notions as point, line, and angle, and little or no use was made of numbers. In the Elements, for example, line segments do not have lengths.

Then in the early part of the 17th century there occurred the greatest advance in geometry since Euclid. It was not the work of one man--such advances seldom, if ever, are. Instead, it occurred when the "intellectual climate" was ready for it. Nevertheless, there was one man whose name is so universally associated with the new geometry that you should know it. That man was René Descartes, a French mathematician and philosopher, who lived from 1596 to 1650. The essential novelty in the new geometry was that it used algebraic methods to solve geometric problems. Thus it brought together two subjects which until then had remained almost independent.

The link between geometry and algebra is forged by coordinate systems. In essence, a coordinate system is a correspondence between the points of some "space" and certain ordered sets of numbers. (We use quotation marks because the space may be a curve, or the surface of a sphere, or some other set of points not usually thought of as a space.) You are already familiar with a number of different coordinate systems, some studied in earlier mathematics courses, others met with in other fields, such as geography. In elementary algebra you introduced coordinates into a plane by drawing two mutually perpendicular lines (axes) in the plane, choosing a positive direction on each and a unit length common to both, and associating with each point the ordered pair of real numbers representing the directed distances of the point from the two axes. The location of a point on the earth's surface is often given in terms of latitude and longitude. An artilleryman sometimes locates a target by saying how far away it is, and in what direction it lies with respect to an arbitrary fixed direction established by setting up an aiming post. This is what is called a polar coordinate system for the plane.

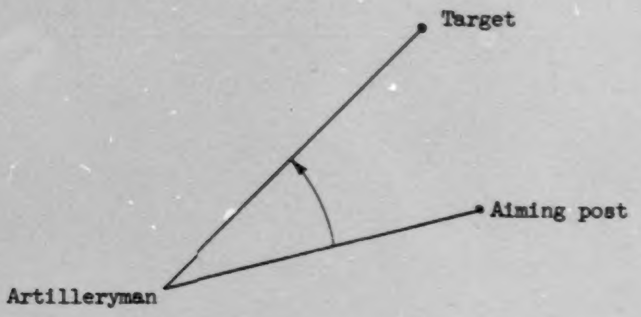


Figure 1-1

A point P on a right circular cylinder could be identified by means of the directed distance z and the measure of the angle θ shown in Figure 1-2.

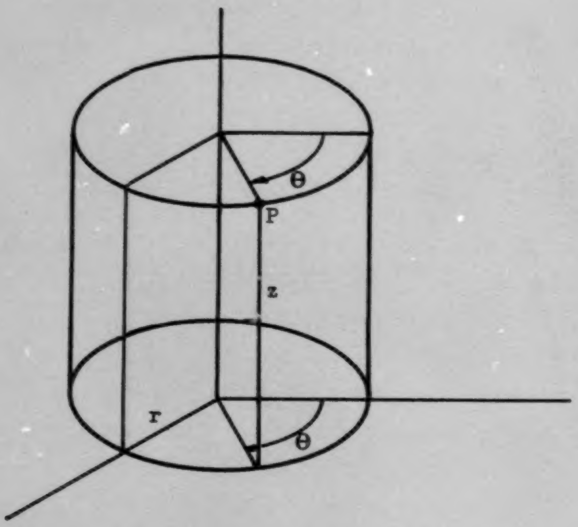


Figure 1-2

If, instead of one right circular cylinder, we consider all such cylinders with the same axis, we can locate any point in space by giving the radius r of the cylinder on which it lies and its z- and θ- coordinates on that cylinder. The result is called a cylindrical coordinate system for space.

A fly on a doughnut (a point on a torus) could be located by means of the measures (in degrees, radians, or any other convenient unit) of the angles θ and ϕ shown in the figure below.

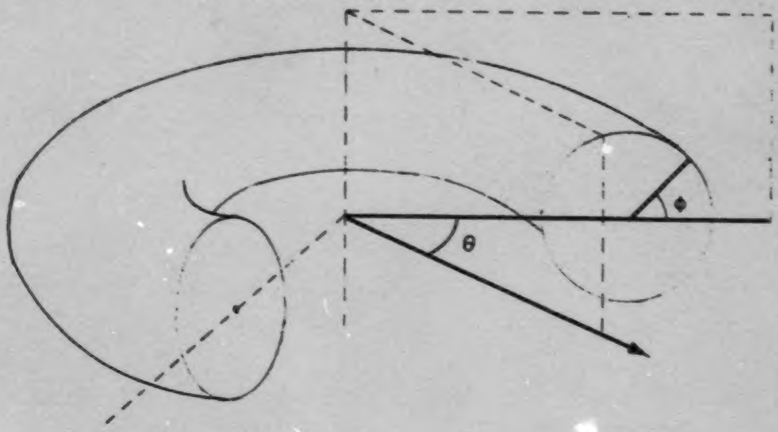


Figure 1-3

The position of an artificial satellite at a certain moment could be specified by giving its vertical distance from the earth's surface (or center) and the latitude and longitude of the point of the earth's surface directly "below" the satellite.

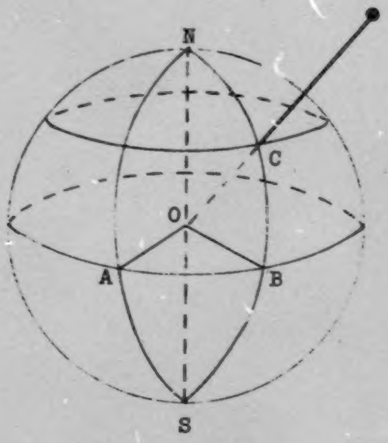


Figure 1-4

The result is called a spherical coordinate system for space.

A coordinate system could be set up even for a "space" which is quite irregular. We may note that your home address is a set of coordinates with which we locate a particular point, your home, relative to the streets and avenues of the town you live in. These streets and avenues, which need not be straight, are the "coordinate lines", and the numbers of the houses on them indicate, in some reasonable way, the positions along these lines.

Once a coordinate system has been established, interesting sets of points can be represented by suitable conditions on their coordinates. The equation

$$2x - y + 4 = 0$$

represents the line through the points $(-1,2)$ and $(2,8)$, where we are using rectangular coordinates. The inequality

$$x^2 + (y - 2)^2 < 9$$

represents the set of points not as far as 3 units distant from $(0,2)$, in other words, the interior of the circle with radius 3 and center $(0,2)$. The equation

$$x^2 - y^2 = 0$$

represents the two lines through the origin making angles of 45° and 135° with the x-axis.

By means of coordinate systems we can, if you like, arithmetize geometry. Problems about geometric figures are replaced by problems about numbers, functions, equations, inequalities, and so forth. Thus one can bring to bear the extensive body of knowledge about algebra, trigonometry, and the calculus which has been developed largely since the 13th century. (In this text we shall use no calculus, but if later you study the subject, you will see that it would have been, in some places, rather useful to us.)

The definition of analytic geometry given above is of the sort found in dictionaries rather than the sort used in mathematics. It tells us not how a technical term will be used in the remainder of this book but how a non-technical phrase is commonly used. As the discussion above indicates, both the subject matter and the methods of this book are already fairly familiar to you. You have even put them together in earlier courses. For example, you know that the graph (in a plane) of an equation of the form

$$ax + by + c = 0 \tag{1}$$

is a straight line, and that the problem of finding the intersection of two lines in a plane can be solved by finding the solution of a system of two equations like (1). You also know that the locus of all the points in a plane which are as far from a fixed line as they are from a fixed point not on that line (this is called a parabola) has an equation of the form

$$y^2 = 4cx$$

if you choose the proper coordinate system. In this book we shall take up many such problems, and by the time you reach the end of it you will have some idea of the power of the new method which Descartes and his contemporaries introduced into geometry.

1-2. Why Study Analytic Geometry?

A chief reason for studying analytic geometry is the power of its methods. Certain problems can be solved more readily, more directly, and more simply by such methods. This is true not only for the problems of geometry and other branches of mathematics, but also for a wide variety of applications in statistics, physics, engineering, and other scientific and technical fields.

Using algebraic methods to solve geometric problems permits easy generalization. A result obtained in one or two dimensions can often be extended at once to three or more dimensions. It is often just as easy to prove a relation in space of n dimensions as it would be in space of two or three dimensions. In fact, much of the work in higher dimensions is essentially algebra with geometric terminology.

Analytic geometry ties together and applies in a new and interesting context what you have been learning about number systems, algebra, geometry, and trigonometry. It should lead to mastery in handling mathematics you have studied previously. As you study this course you will have many opportunities to use knowledge and methods that constitute your present mathematical equipment. You will also learn new methods. Sometimes the new methods will seem awkward or difficult at first when compared with methods you have been using. You should keep in mind that what you are doing is learning about the methods and how to apply them.

As a student, you may at times be directed to use a certain method to gain facility with it. Real problems, whether in mathematics, science, or industry, do not come equipped with a mathematical setting and a prescribed method. By the end of this course you should have a greater variety of mathematical weapons in your arsenal, and more powerful ones. You should be more able then to select effective mathematical weapons to attack problems. Thus another important reason for studying analytic geometry is the value it will have for you in future courses -- not just courses in mathematics but in physics, statistics, engineering, and science in general.

There is a current trend to combine analytic geometry and calculus. When this occurs, much that is of value in the subject of analytic geometry is lost. Because such a course is primarily calculus, only such parts of analytic geometry as are immediately useful in the calculus are kept. By studying a separate course in analytic geometry, you have a better opportunity to understand the coherence of the subject, the diversity of its methods, and the wide variety of problems to which it may be applied.

One of the most important reasons for studying analytic geometry is to gain understanding of the interplay of algebra and geometry. Algebra contributes to analytic geometry by providing a way of writing relationships, a method not only of proving known results but also of deriving previously unknown results. Geometry contributes to algebra by providing a way of visualizing algebraic relations. This visualization, or picture, helps you to understand the algebraic discussion. In the framework provided by a coordinate system, you will do geometry by doing algebra, and see algebra by looking at geometry. Algebra and geometry are intermeshed in analytic geometry; each strengthens and illuminates the other.

Chapter 2

COORDINATES AND THE LINE

2-1. Linear Coordinate Systems.

In our previous study of mathematics we have already encountered at least three major mathematical structures, arithmetic, the algebra of real numbers, and Euclidean geometry. The great German mathematician, David Hilbert (1862-1943), showed that all geometric problems could be reduced to problems in algebra. Our goal here need not be so drastic. We are not trying to eliminate the need for geometry, but rather to establish connections between algebra and geometry. This will enable us to bring to bear on a single problem both the power of algebraic techniques and the structural clarity of geometry.

It turns out that we are able to effect these connections between algebra and geometry by establishing certain one-to-one correspondences between real numbers and points on a line and between real numbers and angles.

In our study of geometry we adopted an important postulate:

The Ruler Postulate. The points of a line can be placed in correspondence with the real numbers in such a way that

(1) To every point of the line there corresponds exactly one real number,

(2) To every real number there corresponds exactly one point of the line, and

(3) The distance between two points is the absolute value of the difference of the corresponding numbers.

We defined such a correspondence to be a coordinate system for the line. We called the number corresponding to a given point the coordinate of the point.

In order to assign a coordinate system to a given line we adopted another postulate:

The Ruler Placement Postulate. Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

We found these postulates to be extremely useful when we defined such concepts as congruence for segments, and order or betweenness for collinear points. We shall want to review and extend these ideas in this text, for it is through coordinate systems that we are able to relate the algebra of numbers to the geometry of sets of points. We shall first extend our notion of a coordinate system.

In our theoretical development of geometry we had no need to mention units; the measure of distance between each pair of points was always a fixed, though unspecified, number. We did not need to know what these numbers were, but only how the measure of distance between one pair of points compared with the measure of distance between a second pair of points. Was the first number as large as the second? Was it larger? Was it twice as large? In applying our theoretical knowledge to specific problems we found that we could use any units we pleased if we were consistent in our usage throughout each given problem. If we did a problem in inches rather than in feet, the numbers we obtained were twelve times as great, but equal distances were still measured by equal numbers. A greater distance had a greater measure, and a shorter distance had a smaller measure, but the ratio of these distances was the same for both choices of unit. Although the measures of distance between pairs of points depended upon the choice of units, within a given problem the measures in one unit were always proportional to the corresponding measures in another unit.

What we discovered in effect was that relative to a given point on a line there are not just two coordinate systems for the line, one oriented in each direction. For each point and each sense of direction on the line there is a coordinate system for the line corresponding to each choice of unit for measuring distance. In each of these coordinate systems the orientation corresponds to one sense of direction for the line and the coordinate of the given point is zero. Since there are infinitely many choices of unit, there are infinitely many coordinate systems for each point and sense of direction on the line.

In this text we are not attempting to develop a rigorous deductive system as we did in geometry. Rather we want to develop and extend the concepts and techniques which we can use to solve problems. Our basic technique will be to introduce coordinate systems. It is so important to utilize the freedom to choose coordinate systems on a line that we state the following guiding principle:

LINEAR COORDINATE SYSTEM PRINCIPLE. There exist coordinate systems for any line such that:

(1) If P and Q are any two distinct points on the line and p and q are any two distinct real numbers, there is a coordinate system in which the coordinate of P is p and the coordinate of Q is q .

(2) If P, Q, R, and S are collinear points with coordinates p, q, r, and s respectively in one coordinate system and p', q', r', and s' respectively in a second coordinate system, if P and Q are distinct, and if R and S are distinct, then

$$\frac{|p' - q'|}{|p - q|} = \frac{|r' - s'|}{|r - s|} .$$

DEFINITION. If a coordinate system on a line assigns the coordinates r and s to the points R and S , then $|r - s|$ is the *measure of distance between R and S relative to the coordinate system.*

This nicety of expression is necessary when we are trying to explain and distinguish concepts which are often confused. As our understanding increases, we may speak more colloquially, and use whatever level of precision is appropriate to the topic and setting. What is important is that a lack of precision should reflect our choice and not our ignorance.

For convenience, and if there is no danger of ambiguity, we shall call this the *distance* between R and S .

We denote the distance between R and S by d(R,S) .

Wherever the context makes clear that only a single coordinate system is being considered, we shall adopt the convention that a is the coordinate point of A, b is the coordinate of point B, c is the coordinate of point C, ... We shall call the point with coordinate zero the *origin* of the coordinate system. The point with coordinate one is called the *unit-point*.

It is sometimes convenient to think of the *directed distance from* R *to* S, which we define to be the number $s - r$. We shall need this idea in the next section.

We shall also find it necessary to use the notion of a *directed segment*, which we define to be the set whose elements are the segment and the ordered pair of its endpoints, or $\{\overline{RS},(R,S)\}$. We shall denote such a directed segment by $\overrightarrow{RS}$. The directed segment $\overrightarrow{RS}$ is said to *emanate* from R and *terminate* in S. However, we should note that directed distance is related to the choice of coordinate system and a directed segment is related to the choice of order for its endpoints. The *length* or *magnitude* of the directed segment $\overrightarrow{RS}$ is the length of $\overline{RS}$, or $d(R,S)$. The ordering of the pair of endpoints (R,S) is related to our intuitive notion of sense of direction, from R to S. We shall find that this alliance of the concepts of magnitude and sense of direction in directed segments is basic to our development of a powerful tool of analysis in Chapter 3.

We conclude with two examples illustrating some of the ideas introduced above.

Example 1. Let us perform a practical experiment. Take a ruler which is marked in inches and another which is marked in centimeters; use each of these rulers to measure the distances between the pairs of labeled points in Figure 2-1. Record your results and compare them.

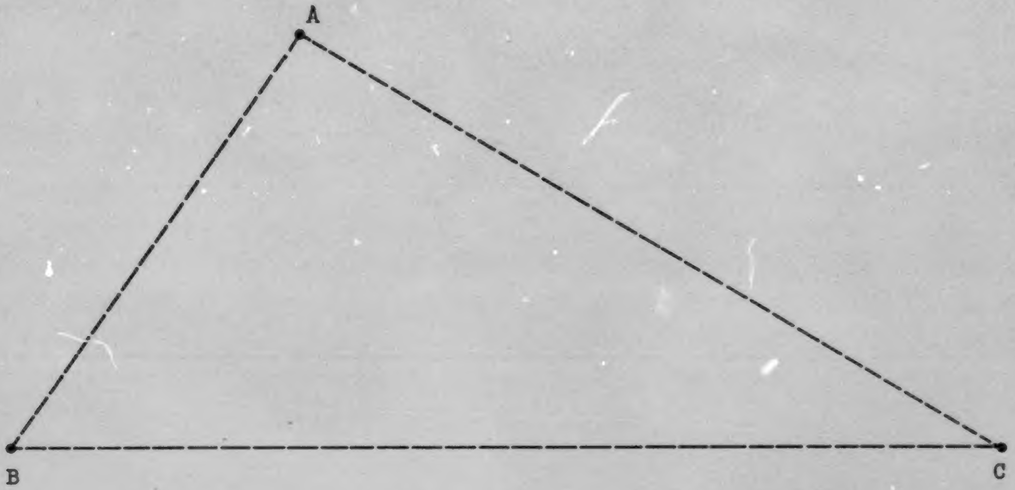


Figure 2-1

Discussion. If a ruler is old or damaged at an end, we prefer not to measure from the end. When we made the measurements required above, we happened to place the unit point of the coordinate system on the inch ruler at A and found that in this case the coordinates of B and C were $3\frac{7}{8}$ and $5\frac{5}{8}$ respectively. When we placed the unit point at B , we found the coordinate of C to be $6\frac{5}{8}$. Since the measure of distance is the absolute value of the difference between the coordinates, we concluded that in inches $d(A,B) = 2\frac{7}{8}$, $d(A,C) = 4\frac{5}{8}$, and $d(B,C) = 5\frac{5}{8}$. We made similar measurements using a ruler marked in centimeter units.

We summarized our measurements in the following table.

Distance	Measure in inches	Measure in centimeters
d(A,B)	$2\frac{7}{8}$	7.3
d(A,C)	$4\frac{5}{8}$	11.7
d(B,C)	$5\frac{5}{8}$	14.3

How do these results compare with yours?

We compared the measures to each other, first in inches and then in centimeters:

$$\frac{d(A,B)}{d(A,C)} = \frac{2\frac{7}{8}}{4\frac{5}{8}} \approx .62 \quad , \qquad \frac{d(A,B)}{d(A,C)} = \frac{7.3}{11.7} \approx .62 \ ,$$

$$\frac{d(A,B)}{d(B,C)} = \frac{2\frac{7}{8}}{5\frac{5}{8}} \approx .51 \quad , \qquad \frac{d(A,B)}{d(B,C)} = \frac{7.3}{14.3} \approx .51 \ ,$$

$$\frac{d(A,C)}{d(B,C)} = \frac{4\frac{5}{8}}{5\frac{5}{8}} \approx .82 \quad , \qquad \frac{d(A,C)}{d(B,C)} = \frac{11.7}{14.3} \approx .82 \ .$$

The accuracy of our results cannot exceed that of our measurements. Within these limitations we found that the ratios of corresponding measures of distance were independent of the units.

Then we compared the measurements in centimeters to those in inches for the same pairs of points and for the perimeter of $\triangle ABC$:

$$d(A,B) : \frac{7.3}{2\frac{7}{8}} \approx 2.54 \ ,$$

$$d(A,C) : \frac{11.7}{4\frac{5}{8}} \approx 2.53 \ ,$$

$$d(B,C) : \frac{14.3}{5\frac{5}{8}} \approx 2.54 ,$$

$$\text{Perimeter of } \triangle ABC : \frac{33.3}{13\frac{1}{8}} \approx 2.54 .$$

Within the limits of accuracy which we could expect, we found that the corresponding measurements in centimeters and in inches were proportional.

Example 2. A straight road 180 miles long connects town A to town B . A driver leaves town A for B at the same instant as another driver leaves town B for A . The drivers travel at the uniform rates of speed, 44 ft. per sec. and 88 ft. per sec. respectively. How soon will they meet?

Discussion. In solving this problem we must make some decisions about units. Some information is given in terms of miles and some in terms of feet. Also we are not told in what units to express the answer. Suppose we try two different approaches. We shall first adopt feet and seconds as the units for distance and time.

(1) We must express 180 miles in feet. The constant of proportionality is 5,280 ft. per mile.
Thus

$$180 \text{ (mi.)} \times \frac{5280}{1} \left(\frac{\text{ft.}}{\text{mi.}}\right) = 950{,}400 \text{ (ft.)} .$$

The inclusion of the name of the unit next to the number of units is a common practice in the physical sciences and engineering. It provides an immediate reminder of the significance of the calculations. Such a practice is called a mnemonic (from the Greek μνασθαι meaning to remember).

We let t represent the number of seconds which will elapse before the two drivers meet. We interpret the problem with the following statement of equality:

$$44t + 88t = 950{,}400 ,$$

which is equivalent to

$$132t = 950{,}400$$

and

$$t = 7{,}200 .$$

The drivers will meet in 7,200 seconds.

This result is such a large number that it may not appeal readily to our intuitive sense of duration of time. We might convert this measure to different units in the hope that the answer will be more intuitively meaningful. If we convert to minutes by dividing by 60, we obtain 120 minutes, which is clearer. If we convert to hours by dividing by 60 again, we obtain 2 hours, which is probably the most satisfactory expression of the answer.

If we are able to anticipate the relative size of the answer, we may be able to choose units which will obviate the need to make changes at the end. In this problem we might well have realized that hours were an appropriate unit for time. We might also have simplified the arithmetic had we used miles as the unit of distance. Our solution would then have been:

(2) We convert the rates of speed to miles per hour. The constants of proportionality are $\frac{1}{5280}$ mile per foot, 60 seconds per minute, and 60 minutes per hour. Thus we obtain

$$44\ (\frac{\text{ft.}}{\text{sec.}}) \times \frac{1}{5280}\ (\frac{\text{mi.}}{\text{ft.}}) \times \frac{60}{1}\ (\frac{\text{sec.}}{\text{min.}}) \times \frac{60}{1}\ (\frac{\text{min.}}{\text{hr.}}) = 30\ (\frac{\text{mi.}}{\text{hr.}})$$

and

$$88\ (\frac{\text{ft.}}{\text{sec.}}) \times \frac{1}{5280}\ (\frac{\text{mi.}}{\text{ft.}}) \times \frac{60}{1}\ (\frac{\text{sec.}}{\text{min.}}) \times \frac{60}{1}\ (\frac{\text{min.}}{\text{hr.}}) = 60\ (\frac{\text{mi.}}{\text{hr.}})\ .$$

We let t represent the number of hours which will elapse before the two drivers meet. We interpret the problem with the statement of equality,

$$30t + 60t = 180\ .$$

This is equivalent to

$$90t = 180$$

or

$$t = 2\ .$$

The drivers will meet in 2 hours.

The first example illustrates the assertions which led to the formulation of the Linear Coordinate System Principle. It also suggests that when we change the coordinate system, we do not lose the notion of congruence for segments, which is defined in the SMSG Geometry on the basis of equal lengths. In the next section we shall see that the concept of order or betweenness is also preserved in linear coordinate systems.

The second example points up the necessity for using units consistently throughout the solution of a problem. It also illustrates the advantages inherent in the freedom to choose the scale or units of a coordinate system.

Exercises 2-1

1. Take a sheet of ordinary lined paper and use a lateral edge to make a "ruler" by assigning coordinates to the ends of the lines. Use this ruler to "measure" Figure 2-1. Following the outline of the discussion in Example 1, compare your measurements to each other and to the measurements in Example 1. Find the constants of proportionality which relate the units of your ruler to inches and centimeters.

2. In Example 1 it was asserted that our results agreed within the limitations of accuracy which might be expected. Show that the accuracy of our results is consistent with the accuracy of our measurements.

   We obtained 2.53 rather than 2.54 as the constant of proportionality relating one measurement in centimeters to the corresponding measurement in inches. Justify that this discrepancy is not significant.

3. Assume that the earth is a sphere of radius 3963 miles. A man of extraordinary powers is able to walk completely around the earth at the equator. During this trip his head is always 6 feet farther from the center of the earth than his feet are. Thus the path of the man's head is longer than the path of his feet. Determine how much longer. Let $\pi = 3.1416$. Try to anticipate the appropriate units for the answer.

4. What is the scale of the map on which the "distance" from New York to San Francisco is shown by a line $7\frac{1}{2}$ inches long?

5. (See Exercises 3 and 4.) A model of the earth, or globe, has a 24-inch diameter. What is the scale of this model? How long on the surface of this model would be the "line" from New York to San Francisco?

6. A bicyclist starts along the road at the rate of 8 miles per hour. Two hours later his friend starts after him on a scooter at the rate of 32 kilometers per hour.

   (a) How far apart are the friends one hour later?

   (b) How long and how far have they traveled when they meet?

7. Two bicyclists start at the same time from points 30 miles apart and ride directly toward each other until they meet. The first rides at 4 miles per hour, the second at 5 miles per hour. At the instant they start a preposterous bee starts from the first bicycle toward the second, flying at an unvarying rate of 10 miles per hour. As soon as he meets the second bicycle, the bee turns back and flies to the first, then back to the second, ... He continues to do so until the two riders meet.

   (a) How long in time and distance was the first leg of the bee's flight?

   (b) What was the total length of the bee's flight in time and distance?

2-2. Analytic Representations of Points and Subsets of a Line.

In this section we confine our attention to a line on which a coordinate system has been chosen. We shall let "a" stand for the coordinate of the point A , "b" for that of B , and so forth.

We shall show that the description of betweenness of points is preserved in any linear coordinate system. We shall also show that conditions on points and subsets of a line may be represented by means of relations involving coordinates.

In the SMSG Geometry we defined the concept of order for three distinct collinear points. The point B is between the points A and C if and only if $d(A,B) + d(B,C) = d(A,C)$. We proved that when B is between A and C , either $a < b < c$ or $a > b > c$; that is, the coordinate of B is between the coordinates of A and C . We also realized that the converse of this theorem is true. Lastly, we used coordinates to deduce that of three distinct collinear points one and only one is between the other two.

If we change to a coordinate system with a different unit, the measures of distance will change, but the Linear Coordinate System Principle assures us that the corresponding new distances will be proportional to the old. If a, b, and c are the original coordinates of three distinct collinear points and a', b', and c' are new coordinates, then

$$\frac{|a' - b'|}{|a - b|} = \frac{|b' - c'|}{|b - c|} = \frac{|a' - c'|}{|a - c|} .$$

If we let the positive real number k represent the equal ratios above, we may write

(1) $\quad |a' - b'| = k|a - b|$, $|b' - c'| = k|b - c|$, and $|a' - c'| = k|a - c|$.

In the original coordinate system we denote the measures of distance between points by $d(A,B)$, $d(B,C)$, and $d(A,C)$; in the new coordinate system we denote the measures by $d'(A,B)$, $d'(B,C)$, and $d'(A,C)$. By definition,

(2) $\quad d(A,B) = |a - b|$, $d(B,C) = |b - c|$, $d(A,C) = |a - c|$,

and

(3) $\quad d'(A,B) = |a' - b'|$, $d'(B,C) = |b' - c'|$, $d'(A,C) = |a' - c'|$.

Now if B is between A and C , then by definition,

$$d(A,B) + d(B,C) = d(A,C) .$$

If we substitute the equal quantities from (2), we obtain

$$|a - b| + |b - c| = |a - c| ,$$

which, since $k \neq 0$, is equivalent to

$$k|a - b| + k|b - c| = k|a - c| .$$

If we substitute the equal quantities from (1) and (3), we obtain first

$$|a' - b'| + |b' - c'| = |a' - c'|$$

and then

$$d'(A,B) + d'(B,C) = d'(A,C) .$$

Thus, the condition describing the order of points on a line is independent of the choice of coordinate system for the line.

Once we have established a criterion for describing the order of points on a line, we are able to define such basic geometric entities as segments and rays. We recall that the segment $\overline{PQ}$ is the set which contains P, Q, and all points between P and Q , while the ray $\overrightarrow{PQ}$ is the union of $\overline{PQ}$ and the set of all points R such that Q is between P and R .

We described the points between P and Q as interior points of the segment $\overline{PQ}$. Since an interior point of a segment divides the segment into two other segments, we sometimes call it an *internal point of division*. We identify a point of division of a segment by stating the ratio of the lengths of the new segments.

DEFINITION. A point of division X is said to *divide the segment* $\overline{PQ}$ *in the ratio* $\frac{c}{d}$ if and only if

$$\frac{d(P,X)}{d(X,Q)} = \frac{c}{d} .$$

If we let p, q, and x represent the coordinates of P, Q, and X in a coordinate system for the line, we may write

$$\frac{|p - x|}{|x - q|} = \frac{c}{d} .$$

Since X is between P and Q, we know that either $p < x < q$ or $p > x > q$. Thus we may remove the absolute value signs to write either

$$\frac{x - p}{q - x} = \frac{c}{d} \quad \text{or} \quad \frac{p - x}{x - q} = \frac{c}{d} ,$$

which implies

$$dx - dp = cq - cx \quad \text{or} \quad dp - dx = cx - cq .$$

These are both equivalent to

$$cx + dx = dp + cq ,$$

$$x = \frac{dp + cq}{c + d} , \tag{4}$$

or

$$x = \frac{d}{c + d} p + \frac{c}{c + d} q . \tag{5}$$

Since c and d are either both positive or both negative, x is always defined in terms of p, q, c, and d.

Equation (4) suggests the description of the coordinate of the point of division as a "weighted average" of the coordinates of the endpoints of the segment. The phrase "weighted average" is suggested by the placement of a fulcrum. When two different weights at the ends of a lever are in balance, the fulcrum is closer to the heavier weight than to the lighter weight. In determining a point of division the heavier "weight" is assigned to the coordinate of the closer point and the lighter "weight" to the coordinate of the more remote point.

Example 1. Express the coordinate of the midpoint of segment $\overline{PQ}$ in terms of p and q, the coordinates of the endpoints.

Solution. By definition the midpoint X of a segment is an interior point equidistant from the endpoints. Thus it is a point of division which divides the segment in the ratio one to one. In this case c and d may both be one, and we may write

$$x = \frac{p + q}{2}$$

or

$$x = \frac{1}{2} p + \frac{1}{2} q .$$

In Equation (5) above the coefficients of p and q add up to one. If we let $\frac{d}{c + d} = a$ and $\frac{c}{c + d} = b$, we may write

$$x = ap + bq \text{ , where } a > 0 \text{ , } b > 0 \text{ , and } a + b = 1 .$$

It is interesting to see what happens here if we omit the requirement that both a and b be positive. Our equation is now

(6) $$x = ap + bq \text{ , where } a + b = 1 .$$

If b is zero, a is one and Equation (6) gives the coordinate of P . If a is zero, b is one and Equation (6) gives the coordinate of Q .

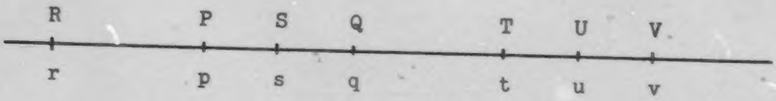


Figure 2-2

In Figure 2-2 we have indicated several points on line $\overleftrightarrow{PQ}$, as well as their coordinates. For convenience let us assume that $r < p < s < q < t < u < v$. We have already seen that if S is the midpoint of $\overline{PQ}$, $s = \frac{1}{2} p + \frac{1}{2} q$; that is, in Equation (6) $a = b = \frac{1}{2}$. Also, p and q are determined by the conditions $a = 1$, $b = 0$ and $a = 0$, $b = 1$ respectively. Let us suppose that $d(P,Q) = d(R,P) = d(Q,T) = d(T,V)$ and that U is the midpoint of $\overline{TV}$. We may determine the coordinates r, t, u, and v in terms of p and q .

The assumption for order of the coordinates permits us to remove the absolute value signs and write:

$$\frac{p - r}{q - r} = \frac{1}{2}, \quad \frac{t - q}{t - p} = \frac{1}{2}, \text{ and } \frac{v - q}{v - p} = \frac{2}{3}$$

which imply

$$r = 2p - q, \quad t = -p + 2q, \text{ and } v = -2p + 3q \text{ respectively.}$$

Since U is the midpoint of $\overline{TV}$,

$$\begin{aligned} u &= \frac{1}{2}t + \frac{1}{2}v \\ &= \frac{1}{2}(-p + 2q) + \frac{1}{2}(-2p + 3q) \\ &= -\frac{3}{2}p + \frac{5}{2}q . \end{aligned}$$

Had we chosen to orient the coordinate system in the opposite direction, we would have obtained the same results.

In every case above the sum of the coefficients of p and q is one. This suggests that any point on the line may be represented by adopting appropriate coefficients in Equation (6). This is true, although we do not prove it here. When a variable is expressed by a form similar to the right side of Equation (6), we say that it is expressed as a linear combination of p and q. We shall have occasion to develop this idea in the next chapter. We may describe our conjecture here by saying that the coordinate of any point on a line may be expressed as a linear combination of the coordinates of two given distinct points on the line.

In view of the restriction on Equation (6), we really need only one variable to represent the coefficients. If we let $t = a$, then $b = 1 - t$ and we may write

(7) $x = tp + (1 - t)q$ where t is any real number.

Thus the variable x is related to the constants p and q by a second variable t. It is clear what x represents; it is the coordinate of a point on the line. We know that t represents a real number and we can see that each value of t determines a unique value of x, but it is not immediately clear what t names or measures. Our primary interest is in the variable x; our interest in t is definitely subordinate. When we express one or more variables in terms of yet another variable, we frequently say that we have a parametric representation. The other variable is called a parameter. We shall want to develop this idea in Chapter 5.

In the present case we see that when $t = 0$, $x = q$; when $t = 1$, $x = p$; and when $t = \frac{1}{2}$, $x = \frac{1}{2}p + \frac{1}{2}q$. This suggests the explanation of the role of t. The Linear Coordinate System Principle assures us that there exists another coordinate system on the line $\overleftrightarrow{PQ}$ in which the coordinate of Q is zero and the coordinate of P is one. A point whose coordinate is represented by t in the latter coordinate system is represented by x in the former coordinate system. The coordinates in the two coordinate systems are related by Equation (7).

We have developed several different ways of describing a point on a line by means of equations involving coordinates. We call such descriptions analytic representations. We now turn to analytic representations of subsets of the line.

In earlier courses you have studied a number of subsets of a line. Among them are the following:

$\overleftrightarrow{AB}$, the line through A and B;

$\overrightarrow{AB}$, the ray whose endpoint is A and which contains B;

$\overline{AB}$, the segment with endpoints A and B.

It is possible to represent these and many other subsets of a line analytically. We consider a number of examples below, and ask you to study others in the exercises. In what follows, when we say that b is between a and c (a, b, and c real numbers), we mean that either $a < b < c$ or $c < b < a$. Then B is between A and C if and only if b is between a and c.

$\overleftrightarrow{AB}$ consists of all points X with any real coordinate x. We can say this in the form

$$\overleftrightarrow{AB} = \{X:\ x \text{ is real}\}$$

or in the form

$$\overleftrightarrow{AB} = \{X:\ x^2 \geq 0\}.$$

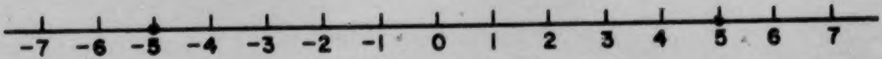
Further

$$\overline{AB} = \{X:\ a \leq x \leq b \text{ or } b \leq x \leq a\}$$

$$\overrightarrow{AB} = \{X:\ b > a \text{ and } x \geq a, \text{ or } b < a \text{ and } x \leq a\}.$$

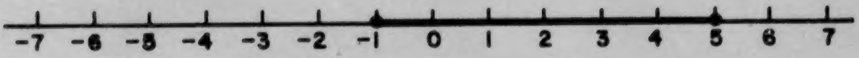
There are two related problems which crop up frequently in analytic geometry, one of which is illustrated above. A set S of points may be specified by geometric conditions, and we may ask for an analytic condition satisfied by the coordinates of points of S but not by those of any other points. On the other hand, we may be given an analytic condition and want to know what points have coordinates satisfying it. You have met both these problems before. The analytic condition was usually an equation, but you have also considered inequalities, and some of the conditions considered below involve other relations. When a set of points consists of those points whose coordinates satisfy a certain condition, we call the set the graph (or locus) of the condition; we call the condition a condition for (or of) the set. These ideas prove more interesting and more important in a plane and in space, but we shall discuss some examples on a line and ask you to work on others.

Example 1. The graph of $|x| = 5$, which is also the graph of $x^2 = 25$, is the set of points with coordinates ± 5.



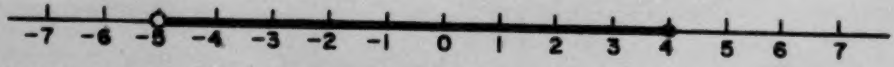
This illustrates the fact that there may be different conditions for the same set of points. (Of course this raises the question of whether the conditions are really different, but at least they were expressed differently.)

Example 2. To find the graph of $|3x - 6| \leq 9$, we observe that $|3x - 6| \leq 9$ is equivalent to $3|x - 2| \leq 9$, or $|x - 2| \leq 3$. The graph is shown below.



The use of the absolute value in measuring distance is an aid in finding the graph. Thus, the graph of the solution set of $|x - 2| \leq 3$ may be interpreted as "the set of all points of the line whose distance from the point with coordinate 2 is less than or equal to 3."

Example 3. Find an analytic condition for the set of points shown below.



(The heavy dot is a device for indicating that the right endpoint is in the set.) An analytic condition for this set is

$$-5 < x \leq 4 \ .$$

Example 4. Let the coordinates of points O , A , X , be 0 , a , x , respectively. Find all points X such that $2d(O,X) + 3d(X,A) = d(O,A)$.

Solution. For any X , $d(O,X) + d(X,A) \geq d(O,A)$. Then, unless $d(O,X) = d(X,A) = 0$, we have

$$2d(O,X) + 3d(X,A) > d(O,A) \ .$$

Thus there is no solution unless O = X = A .

Exercises 2-2

1. Represent graphically:

   (a) $r^2 = 4$

   (b) $(x - 3)^2 = 4$

   (c) $|r - 3| = 2$

   (d) $x + 3 < 7$

   (e) $5 \leq 2 - x$

   (f) $|t + 3| < 3$

   (g) $x(x - 1) > 0$

   (h) $(x - 1)(x + 2) \leq 0$

   (i) $x^2 + 4 < -4x$

   (j) $|2y - 4| = 6$

   (k) $|x - 1.123| < .456$

   (l) $|2s + 2| < 4$

   (m) $|3x + 2| = 1$

   (n) $\sin x\pi = 0$

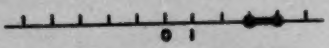
   (o) $2 \sin x\pi = 1$

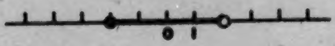
   (p) $\cos \theta > 0$

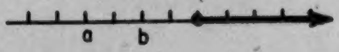
   (q) $|x - a| < \delta$, where $a = 2.35$ and $\delta = 0.44$

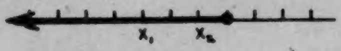
   (r) $|x - a| < \delta$, where $a = 0.44$ and $\delta = 2.35$

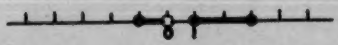
2. Represent analytically:

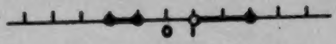
(a) 

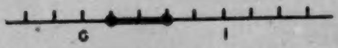
(b) 

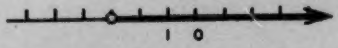
(c) 

(d) 

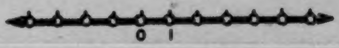
(e) 

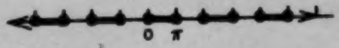
(f) 

(g) 

(h) 

(For Parts (i) and (j) assume the same pattern throughout the line.)

(i) 

(j) 

3. Points O, U, A, and X have coordinates 0, 1, a, and x respectively. Find all values of x that satisfy each of the following conditions:

(a) $d(O,X) = 3d(O,A)$

(b) $d(O,X) + d(U,X) = d(O,U)$.

4. If P and Q have the coordinates given, and if M, A, and B are the midpoint and the two trisection points of $\overline{PQ}$ respectively, find, in each case, the coordinates m, a, and b :

(a) $p = 3$, $q = 12$

(b) $p = -2$, $q = 13$

(c) $p = r + s$, $q = r - s$

(d) $p = (r + t) - 2$, $q = (r + t) + 4$

(e) $p = 2r$, $q = 3t$

(f) $p = 2r + 3s$, $q = 3r - 2s$

(g) $p = r^2 - r$, $q = s^2 - s$

(h) $p = r$, $q = s$.

5. In the equation of the line $\overleftrightarrow{PQ}$

$$x = ap + bq \text{ , where } a + b = 1 \text{ ,}$$

x , p , and q are the coordinates of the points X , P , and Q respectively.

Find the relative positions of X , P , and Q if

(a) $a = 0$

(b) $a = 1$

(c) $0 < a < 1$

(d) $a < 0$

(e) $a > 1$

(f) $b > 1$.

6. In the equation of the line $\overleftrightarrow{PQ}$,

$$x = tp + (1 - t)q \text{ , where } t \text{ is real,}$$

x , p , and q are the coordinates of the points X , P , and Q respectively. For what value(s) of t is

(a) $d(P,X) = 2d(Q,X)$

(b) $2d(P,X) = d(Q,X)$

(c) $d(X,P) = 2d(P,Q)$

(d) $d(P,Q) = d(Q,X)$

Exercises 7-10 are based upon the following situation:

Points A , B , C , D , and E are on the edge of an ordinary 12 inch ruler at positions corresponding to 1 , $1\frac{1}{2}$, $2\frac{1}{2}$, $4\frac{1}{2}$, and 9 respectively. These numbers are the inch-coordinates a , b , c , d , and e , of the corresponding points.

7. Find the ratios (a) $\frac{d(A,B)}{d(B,C)}$, (b) $\frac{d(B,C)}{d(C,D)}$, and (c) $\frac{d(C,D)}{d(D,E)}$.

8. Express

(a) b as a linear combination of a and c .

(b) c as a linear combination of b and d .

(c) d as a linear combination of c and e .

9. Find the inch-coordinates of the trisection points of $\overline{AC}$; of $\overline{BD}$; of $\overline{CE}$.

10. Find the inch-coordinates of points P , Q , and R such that

$$\frac{d(A,B)}{d(B,P)} = \frac{2}{3} \text{ , } \frac{d(B,C)}{d(C,Q)} = \frac{2}{3} \text{ , and } \frac{d(C,D)}{d(E,R)} = \frac{2}{3} \text{ .}$$

2-3. Coordinates in a Plane.

You will recall that the points of a plane can be put into one-to-one correspondence with the ordered pairs of real numbers in the following way. Any two perpendicular lines in the plane are selected as reference lines or *axes*. They are called the *x-axis* and the *y-axis*. The intersection of these lines is called the *origin* and denoted by O . On each axis we use a coordinate system with 0 as origin. Normally the two coordinate systems should use the same units. It is possible to use different coordinate systems on the two axes, but this introduces complications, a few of which will be considered in exercises. If P is any point in the plane, let a and b be the coordinates of the projections of P onto the x-axis and y-axis respectively. Then to P we assign the ordered pair (a,b) of real numbers (*rectangular coordinates*). The first is called the *x-coordinate* or *abscissa* of P , the second the *y-coordinate* or *ordinate* of P . Conversely, if (a,b) is an ordered pair of real numbers, there is a unique point P with abscissa a and ordinate b . It is the intersection of the line perpendicular to the x-axis at the point on that axis with coordinate a , and the line perpendicular to the y-axis at the point on that axis with coordinate b .

In sketches it is customary, though not necessary, to show the x-axis horizontal with its positive half to the right, the y-axis vertical with its positive half upward. In all sketches we place an x by the end of the line representing the positive half of the x-axis and a y by the end of the line representing the positive half of the y-axis. This is essential when we do not indicate the coordinates of any points on the axes.

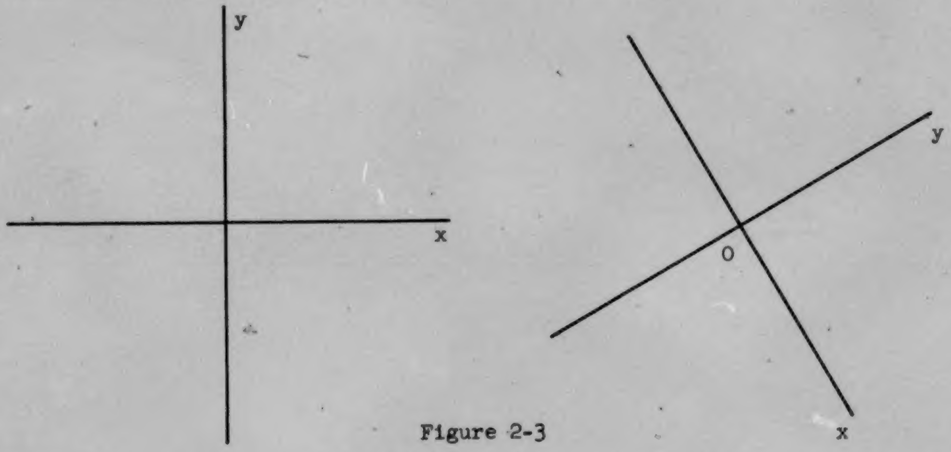


Figure 2-3

We customarily reserve the letter 0 to represent the origin, but do not always include it on a sketch unless we refer to it.

You will also recall that if $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$, then the distance between the two points is

$$d(P_0,P_1) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}\ .$$

We turn now to the problem of expressing the coordinates of any point $P = (x,y)$ of the line L determined by the distinct points $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$ in terms of the coordinates of P_0 and P_1. Let us assume for the time being that $x_0 \neq x_1$ and $y_0 \neq y_1$.

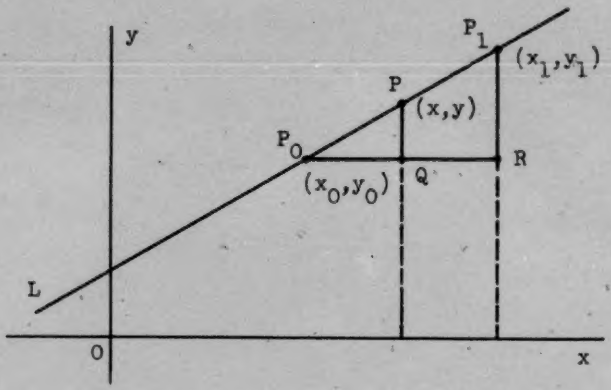


Figure 2-4

In Figure 2-4 $\overleftrightarrow{P_0Q}$ is perpendicular to the y-axis, $\overleftrightarrow{PQ}$ and $\overleftrightarrow{P_1R}$ to the x-axis. Then triangles P_0QP and P_0RP_1 are similar, and hence

$$(1) \qquad \frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}\ .$$

Be sure that you see that the same equation holds if the order of P_0, P_1, and P is different.

If the point P is an internal point of division which divides the segment $\overline{P_0P_1}$ in the ratio $\frac{c}{d}$, then each member of Equation (1) is equal to $\frac{c}{c + d}$ and we may write

$$\frac{x - x_0}{x_1 - x_0} = \frac{c}{c + d} \quad \text{and} \quad \frac{y - y_0}{y_1 - y_0} = \frac{c}{c + d}\ .$$

If we solve these equations for x and y, we obtain

$$x = \frac{dx_0 + cx_1}{c + d} \quad \text{and} \quad y = \frac{dy_0 + cy_1}{c + d}, \tag{2}$$

in which the coordinates of the point of division are expressed as weighted averages of the coordinates of the endpoints of the segment.

We are now in a position to follow exactly the same development as in Section 2-2.

If P is the midpoint of $\overline{P_0P_1}$, it divides the segment in the ratio one to one. In this case we may let $c = d = 1$ and write

$$x = \frac{x_0 + x_1}{2} \quad \text{and} \quad y = \frac{y_0 + y_1}{2}.$$

If in Equations (2) we let $a = \frac{d}{c + d}$ and $b = \frac{c}{c + d}$, we may write

$x = ax_0 + bx_1$ and $y = ay_0 + by_1$, where $a > 0$, $b > 0$, and $a + b = 1$.

If we omit the requirement that a and b be positive, we obtain

$$x = ax_0 + bx_1 \quad \text{and} \quad y = ay_0 + by_1, \quad \text{where } a + b = 1. \tag{3}$$

An analysis similar to that of Equation (6) in the previous section would suggest that each point $P = (x,y)$ on $\overleftrightarrow{P_0P_1}$ corresponds to a unique choice of numbers for a and b in Equations (3), and conversely each pair (a,b) in Equations (3) corresponds to a unique point on $\overleftrightarrow{P_0P_1}$. Thus the x-coordinate of a point on a line may be represented by a linear combination of the x-coordinates of two given distinct points on the line, while the y-coordinate is represented by the same linear combination of the y-coordinates of the given points.

Lastly, we recognize that, because of the restriction on the coefficients in Equations (3), one variable will suffice. If we let $t = b$, then $a = 1 - t$ and we obtain

$$x = (1 - t)x_0 + tx_1 \quad \text{and} \quad y = (1 - t)y_0 + ty_1$$

or

$$\begin{aligned} x &= x_0 + t(x_1 - x_0) \\ y &= y_0 + t(y_1 - y_0) \end{aligned} \quad \text{where } t \text{ is real.} \tag{4}$$

This is a parametric representation of the point $P = (x,y)$ on the line $\overleftrightarrow{P_0P_1}$, where $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$. As we shall see in Chapter 5, this representation is not only useful; for certain problems it is essential. As we observed in the previous section, the parameter t represents the coordinate of P in the linear coordinate system with origin P_0 and unit-point P_1.

The coordinate system for a plane which we have described and used above is called a rectangular or Cartesian coordinate system. The name "Cartesian" comes from Descartes, who is credited with being the first to introduce the theory of algebra into the study of geometry.

Exercises 2-3

1. If P and Q have the coordinates given, and if M, A, and B are the midpoint and the two trisection points of $\overline{PQ}$ respectively, find the coordinates of M, A, and B in each case:

   (a) $P = (0,0)$, $Q = (6,9)$
   (b) $P = (2,3)$, $Q = (8,12)$
   (c) $P = (5,12)$, $Q = (6,-7)$
   (d) $P = (4,-3)$, $Q = (-9,10)$
   (e) $P = (-6,-3)$, $Q = (6,3)$
   (f) $P = (-3,-6)$, $Q = (-6,-3)$
   (g) $P = (p_1,p_2)$, $Q = (q_1,q_2)$
   (h) $P = (2s,5t)$, $Q = (s,-2t)$
   (i) $P = (4r + 2s, -3r + s)$, $Q = (-r - s, -r - 2s)$

2. Let $P = (x,y)$ be a point on line $\overleftrightarrow{P_0P_1}$, where $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$. Express x as a linear combination of x_0 and x_1 and y as the same linear combination of y_0 and y_1 in the following cases:

   (a) $P_0 = (2,3)$, $P_1 = (6,1)$
   (b) $P_0 = (-4,5)$, $P_1 = (2,-7)$
   (c) $P_0 = (-3,-6)$, $P_1 = (-6,4)$

3. Let $P = (x,y)$ be a point on line $\overleftrightarrow{P_0P_1}$, where $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$. In the following cases express coordinates of P by a parametric representation. Choose the parameter t so that $(x,y) = P_0$ when $t = 0$ and $(x,y) = P_1$ when $t = 1$.

   (a) $P_0 = (2,3)$, $P_1 = (6,1)$

   (b) $P_0 = (-4,5)$, $P_1 = (2,-7)$

   (c) $P_0 = (-3,-6)$, $P_1 = (-6,4)$

4. In the development of Equation (1) in Section 2-3, we assumed that $x_0 \neq x_1$ and $y_0 \neq y_1$. If $x_0 = x_1$ or $y_0 = y_1$, this equation does not hold, but Equation (2) in Section 2-3 does apply. Consequently, the rest of the development is valid in either of these cases. Justify that Equation (2) applies when the conditions are relaxed. [Hint: Show that the problem reduces to the situation discussed in Section 2-2.]

5. Apply the condition given by Equation (1) to decide whether the points A, B, and C with the coordinates given, are collinear. How can you use the formula for the distance between two points to determine whether three points are collinear? Use this method to check your answers.

   (a) $A = (7,0)$, $B = (-3,-6)$, $C = (22,9)$

   (b) $A = (-1,4)$, $B = (3,-14)$, $C = (-5,-6)$

6. For what value of h is the point $P = (h,-3)$ on the line determined by $A = (1,-1)$ and $B = (4,7)$?

2-4. Polar Coordinates.

A rectangular coordinate system is certainly the most frequently employed coordinate system, but it is not always the best choice for a given problem.

The rectangular coordinate system is based upon a grid composed of two mutually perpendicular systems of evenly spaced parallel lines in a plane. An alternative is the polar coordinate system, which is based upon a grid composed of a system of concentric circles and a system of rays emanating from the common center of the circles.

The paths from one point to another on a rectangular grid usually entail motion along two adjacent sides of a rectangle, but the natural paths of physical objects are usually more direct. A football player does not pass the ball to follow the deceptive path of a receiver. Rather he looks for the receiver in a certain area. If he finds the receiver uncovered, he will try to pass the ball just so far in the direction of the receiver. To apply this idea in the plane we require a frame of reference. The frame of reference consists of a fixed point O, called the pole, and a fixed ray $\overrightarrow{OM}$, called the polar axis. The ray has the non-negative part of a linear coordinate system with the origin at O. The position of a point P is uniquely determined by r and θ, its polar coordinates (Figure 2-5a).

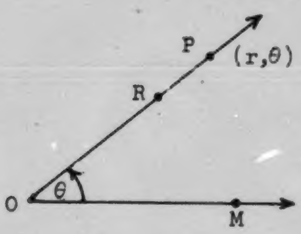


Figure 2-5a

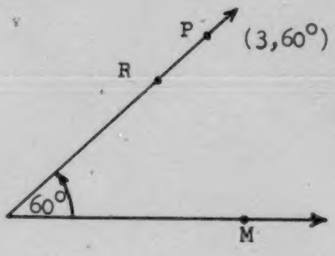


Figure 2-5b

The polar angle θ is an angle generated by rotating a ray $\overrightarrow{OR}$ about O from $\overrightarrow{OM}$ in either direction as far as desired and terminating the rotation in a position such that the line $\overleftrightarrow{OR}$ contains P. If we rotate $\overrightarrow{OR}$ in a counterclockwise direction, $\angle\theta$ has a positive measure; if $\overrightarrow{OR}$ is rotated clockwise, then $\angle\theta$ has a negative measure.

DEFINITION. If $\overrightarrow{OR}$ contains P, then the polar distance $r = d(O,P)$; if P lies on the ray opposite to $\overrightarrow{OR}$, then $r = -d(O,P)$.

Commonly used units of measure for polar angles are degrees and radians. When the usual symbols for numerical measure of angles in degrees, minutes and seconds are omitted, it is understood that radian measure is intended.

The polar coordinates of a point are written as an ordered pair (r,θ), where r is the polar distance and θ is a measure of the polar angle. If the angle is measured in degrees, the symbolism alone indicates that the ordered pair represents polar coordinates. If the measure of the angle is given in radians, the ordered pair of real numbers is indistinguishable from the notation used in rectangular coordinates. If the context does not make clear that these are polar coordinates, we must say so explicitly. If no indication is given, we shall assume that rectangular coordinates are intended.

The pole is a special point. When $r = 0$, the pole is described. In this case $\angle\theta$ may have any measure. $(0,0)$, $(0,60^\circ)$, $(0,180^\circ)$, and $(0,\frac{\pi}{2})$ are all names for the pole. We usually write $(0,\theta)$ to indicate that θ may be any number. The pole is not the only point whose representation is not unique.

A rectangular coordinate system establishes a one-to-one correspondence between points in a plane and ordered pairs of real numbers. It is important to observe that a polar coordinate system does not. In polar coordinates each ordered pair corresponds to a unique point in the plane, but each point is represented by infinitely many ordered pairs of numbers.

For example, some other coordinates for the point P shown in Figure 2-5b are $(3,420^\circ)$, $(3,-300^\circ)$, and $(-3,-\frac{2}{3}\pi)$. See Figure 2-6.

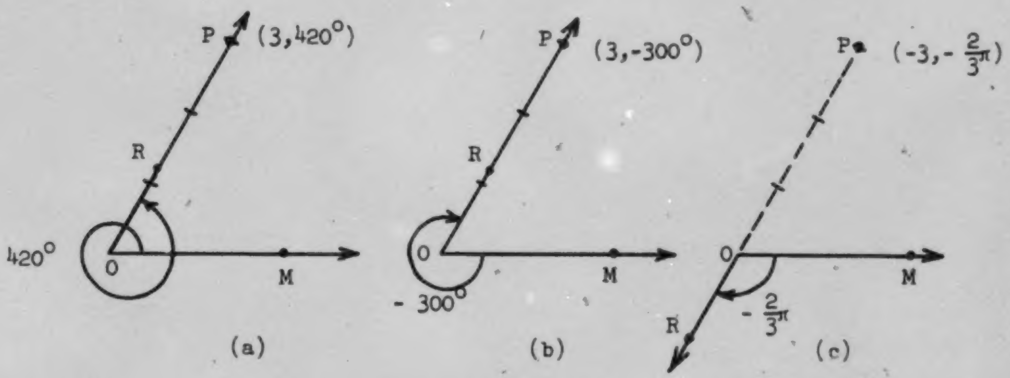


Figure 2-6

In subsequent figures we shall delete the arrowhead from all rays except the polar axis.

The lack of a one-to-one correspondence between points and ordered pairs of numbers necessitates care when we use polar coordinates, but the advantages are sometimes great indeed. For example, the figures which we have used here may remind you of the figures which illustrated the definitions of the trigonometric or circular functions. As you will discover in subsequent chapters, the analytic representations of these functions and allied relations are often simpler in polar coordinates.

Example 1. Plot the points A , B , C , and D , which have polar coordinates $(2,45^{\circ})$, $(3,-120^{\circ})$, $(1,\frac{\pi}{3})$, and $(-2,-\frac{\pi}{4})$ respectively.

Solution.

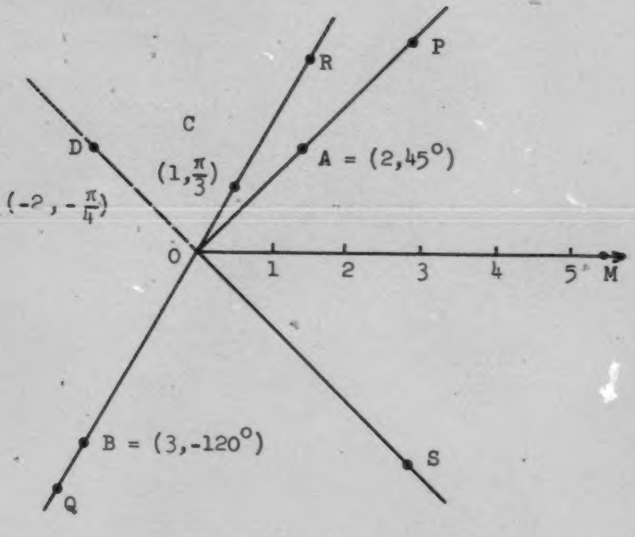


Figure 2-7

Since a measure of $\angle POM = 45^{\circ}$, A is the point on $\overrightarrow{OP}$ such that $d(O,A) = 2$. A measure of $\angle QOM = -120^{\circ}$ and B is the point on $\overrightarrow{OQ}$ such that $d(O,B) = 3$. A measure of $\angle ROM = \frac{\pi}{3}$ and C is the point on $\overrightarrow{OR}$ such that $d(O,C) = 1$. Lastly, a measure of $\angle SOM = -\frac{\pi}{4}$, but since the polar distance is negative, D is the point on the ray opposite to $\overrightarrow{OS}$ such that $d(O,D) = 2$.

Example 2. Find four pairs of polar coordinates, two in degrees and two in radians, for each of the points A , B , and C in Figure 2-8.

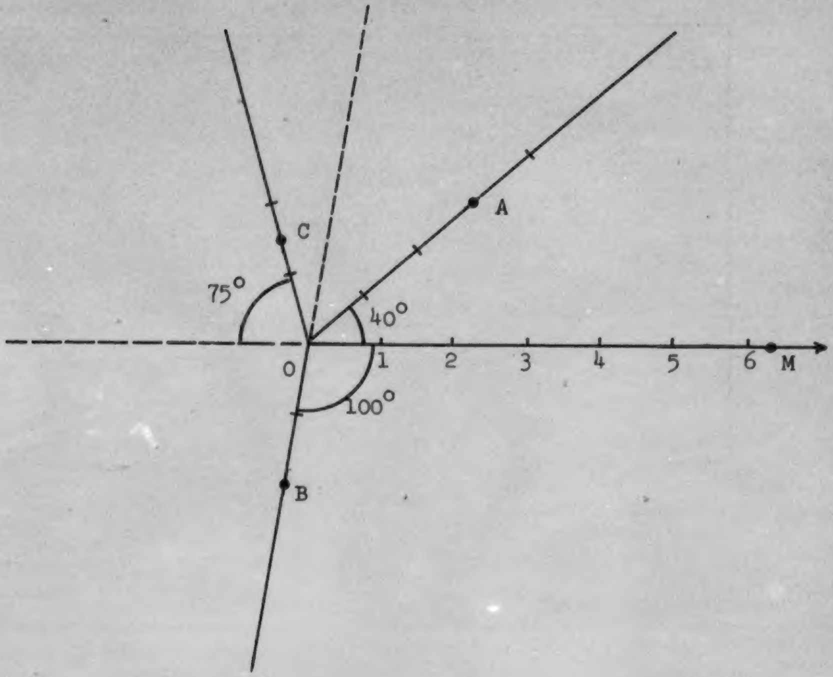


Figure 2-8

Solution. A simple representation for A is $(3,40^\circ)$, but we may also use $(3,-320^\circ)$, $(3,\frac{2\pi}{9})$, and $(-3,\frac{11\pi}{9})$. (There are others, of course.) $B = (2,-100^\circ)$, $(-2,80^\circ)$, $(2,\frac{-5\pi}{9})$, and $(-2,\frac{4\pi}{9})$. $C = (1\frac{1}{2},105^\circ)$, $(1\frac{1}{2},465^\circ)$, $(1\frac{1}{2},\frac{7\pi}{12})$, and $(-1\frac{1}{2},\frac{19\pi}{12})$.

We mentioned that any pair of perpendicular lines in a plane may be chosen as the reference axes for a rectangular coordinate system. Any ray in a plane may be chosen for the polar axis in introducing a polar coordinate system. When we are solving a problem using coordinates, this freedom enables us to choose a system which will simplify the computation. Because we wish to keep this in mind, we state the following principle:

COORDINATE PLANE PRINCIPLE. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two perpendicular lines intersecting at O $(O \neq A$ and $O \neq C)$, there exists a rectangular coordinate system in the plane of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ such that

(i) $\overleftrightarrow{AB}$ is the x-axis, $\overleftrightarrow{CD}$ is the y-axis

and

(ii) in the coordinate systems on the axes, the coordinates of A and C are positive.

In any plane containing the ray $\overrightarrow{OM}$ there exists a polar coordinate system such that $\overrightarrow{OM}$ is the polar axis.

In some situations we must use both rectangular and polar coordinate systems in the same plane. Usually we let the polar axis coincide with the non-negative half of the x-axis. Coordinates in both systems are assigned to each point in the plane, but we shall need equations relating the coordinates in order to change back and forth.

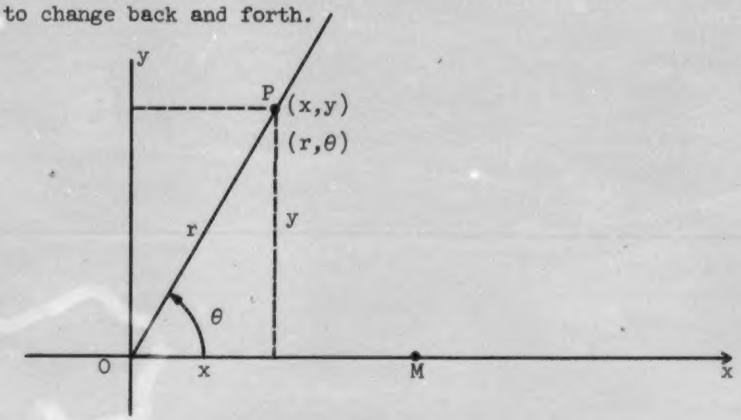


Figure 2-9

In Figure 2-9, we see that

(1)
$$x = r\cos\theta$$
$$y = r\sin\theta$$

and

(2)
$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}, \text{ where } x \neq 0 .$$

In Equations (2) we note that, as we might have expected, r and θ are not uniquely defined. You should verify these equations for points in other quadrants.

We may use Equations (1) to transform from polar coordinates to rectangular coordinates and Equations (2) to find polar coordinates for points whose rectangular coordinates are known.

Example 3. Find the rectangular coordinates of the point designated in polar form by $(8, -60^\circ)$.

Solution.

$$x = 8 \cos(-60^\circ) = 8(\tfrac{1}{2}) = 4$$

$$y = 8 \sin(-60^\circ) = 8(-\tfrac{1}{2}\sqrt{3}) = -4\sqrt{3}$$

Example 4. Find a polar representation for the point with rectangular form $P = (-2,-2)$.

Solution. $r^2 = (-2)^2 + (-2)^2 = 8$; therefore, $r = \pm 2\sqrt{2}$. Also, $\tan\theta = \frac{-2}{-2} = 1$; hence, $\theta = \frac{\pi}{4} + n\pi$, n an integer . It is necessary to match the values of r and θ which correctly locate P . For example, $(2\sqrt{2}, \frac{\pi}{4})$ is not a correct solution, as these coordinates locate a point in the first quadrant. But $(2\sqrt{2}, \frac{5\pi}{4})$ and $(-2\sqrt{2}, \frac{\pi}{4})$ are two of the possible correct designations for P .

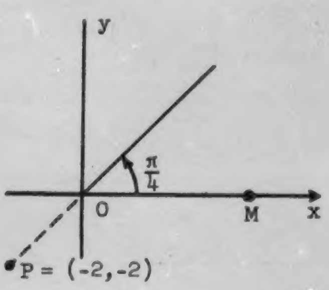


Figure 2-10

Example 5. Find the distance between the points P_1 and P_2 whose polar coordinates are (r_1, θ_1) and (r_2, θ_2) respectively.

Solution. We have an expression for the distance between two points in terms of their rectangular coordinates,

$$(3) \qquad d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \,.$$

We may use this expression if we transform the coordinates of P_1 and P_2 to rectangular form. We use Equations (1) to obtain

$$x_1 = r_1 \cos \theta_1 \,, \; y_1 = r_1 \sin \theta_1$$

$$x_2 = r_2 \cos \theta_2 \,, \; y_2 = r_2 \sin \theta_2 \,.$$

We square both members of Equation (3) and substitute these values to obtain

$$\big(d(P_1, P_2)\big)^2 = (r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2$$

or

$$\big(d(P_1, P_2)\big)^2 = r_2^2 \cos^2 \theta_2 - 2r_1 r_2 \cos \theta_2 \cos \theta_1 + r_1^2 \cos^2 \theta_1$$
$$+ r_2^2 \sin^2 \theta_2 - 2r_1 r_2 \sin \theta_2 \sin \theta_1 + r_1^2 \sin^2 \theta_1 \,.$$

If we apply the distributive and other laws, this becomes

$$\big(d(P_1, P_2)\big)^2 = r_1^2(\cos^2 \theta_1 + \sin^2 \theta_1) + r_2^2(\cos^2 \theta_2 + \sin^2 \theta_2)$$
$$- 2r_1 r_2(\cos \theta_2 \cos \theta_1 + \sin \theta_2 \sin \theta_1) \,.$$

Now $\qquad \cos^2 \theta_1 + \sin^2 \theta_1 = 1 \,, \; \cos^2 \theta_2 + \sin^2 \theta_2 = 1 \,,$

and $\qquad \cos \theta_2 \cos \theta_1 + \sin \theta_2 \sin \theta_1 = \cos(\theta_2 - \theta_1) \,.$

We substitute these equivalent values to obtain

$$(3) \qquad \big(d(P_1, P_2)\big)^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1) \,.$$

We might have obtained this expression directly by applying the Law of Cosines to triangle OP_1P_2 in Figure 2-11.

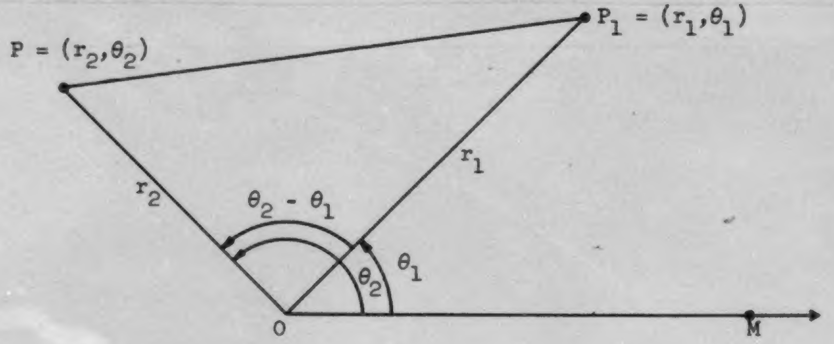


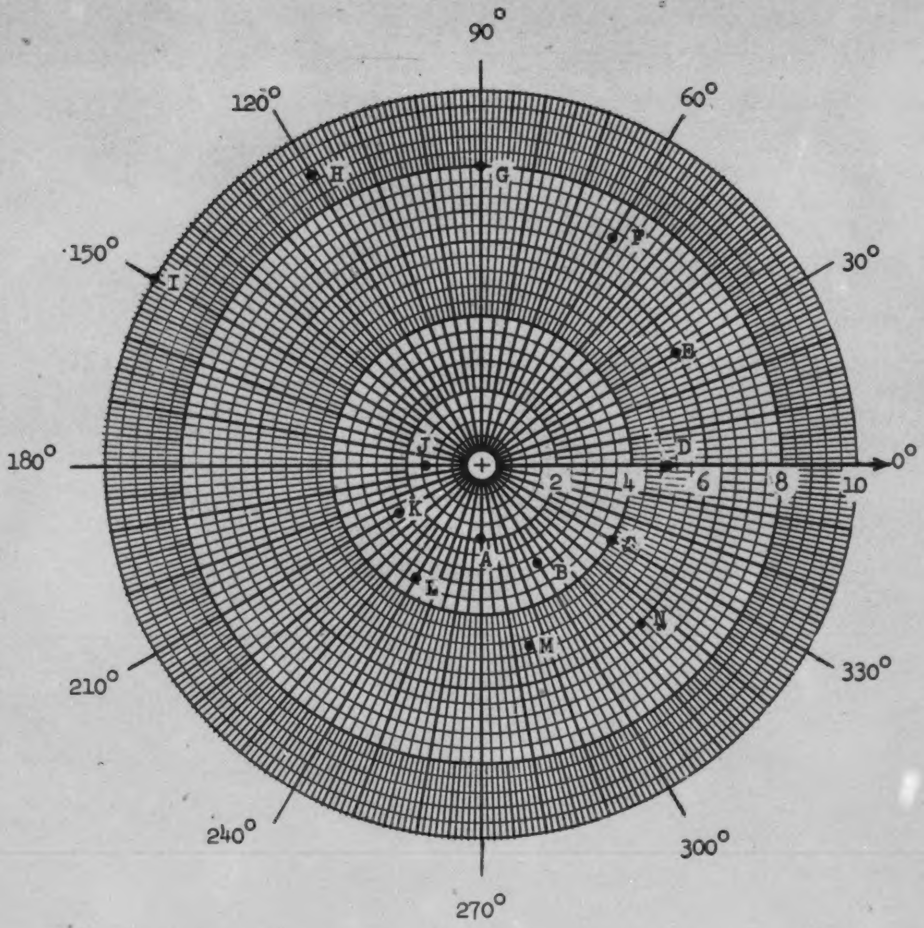
Figure 2-11

Thus the distance formula in polar coordinates is an application of the Law of Cosines.

Exercises 2-4

1. Plot the following points and for each list three pairs of coordinates: $(5,135^\circ)$, $(2,90^\circ)$, $(-4,45^\circ)$, $(3,-120^\circ)$.
2. Plot the points whose polar coordinates are $(-2,45^\circ)$, $(-4,210^\circ)$, $(3,2)$, $(-3,-\frac{3}{2})$, $(4,0^\circ)$, $(0,\frac{\pi}{2})$, $(-4,180^\circ)$.
3. Plot the vertices of an equilateral triangle, the centroid coincident with the pole and a vertex on the polar axis, and give polar coordinates of the vertices.
4. Draw graphs representing the set of points $\{(r,\theta) : r = 4\}$; the set of points $\{(r,\theta) : \theta = 45^\circ\}$.

5.


90°
120°
60°
150°
30°
180°
0°
210°
330°
240°
300°
270°
G
H
F
I
E
J
D
2
4
6
8
10
K
C
A
B
L
N
M

For each of the points indicated on the preceding diagram give five pairs of polar coordinates; in the first pair have $r > 0$, and $0^\circ \le \theta < 360^\circ$, in the second pair have $r > 0$, and $-360^\circ < \theta \le 0^\circ$, in the third pair have $r < 0$, and $0^\circ \le \theta < 360^\circ$, in the fourth pair have $0 \le \theta < \pi$, in the fifth pair have $0^\circ \le \theta < 180^\circ$.

6. Find the rectangular representation of the points whose polar coordinates are

(a) $(0,90^\circ)$

(b) $(\sqrt{2},-45^\circ)$

(c) $(5,420^\circ)$

(d) $(4,0^\circ)$

(e) $(1,\pi)$

(f) $(\sqrt{2},\frac{5\pi}{2})$

(g) $(-2,\frac{1}{3}\pi)$

(h) $(2,-\frac{\pi}{4})$

7. Write a polar representation for the points whose rectangular coordinates are

(a) $(1,1)$

(b) $(2,-2)$

(c) $(p,0)$

(d) $(0,q)$

(e) $(-\sqrt{3},1)$

(f) $(-1,-\sqrt{3})$

(g) $(5,2)$

(h) $(-4,1)$

8. Use polar coordinates to find the distance between the points A and B. Then change to rectangular coordinates and verify your result.

(a) $A = (2,150^\circ)$, $B = (4,210^\circ)$

(b) $A = (5,\frac{5}{4}\pi)$, $B = (12,\frac{7}{4}\pi)$

9. Find the distance between each of the following pairs of points.

(a) $A = (3,0^\circ)$, $B = (5,90^\circ)$

(b) $A = (2,37^\circ)$; $B = (3,100^\circ)$

(c) $A = (6,100^\circ)$, $B = (8,400^\circ)$

(d) $A = (-1,45^\circ)$, $B = (3,165^\circ)$

(e) $A = (3,20^\circ)$, $B = (5,140^\circ)$

(f) $A = (5,-60^\circ)$, $B = (10,-330^\circ)$

10. On a polar graph chart such as in Exercise 5 construct a hexagon with vertices $(10,0^\circ)$, $(10,60^\circ)$, etc. Then construct all its diagonals and write the coördinates of all their intersections (other than the pole).

11. Let (r_0,θ_0) represent a point P . Find general expressions for all the possible polar coordinates of P

(a) when θ_0 is in degrees and

(b) when θ_0 is in radians.

2-5. Lines in a Plane.

Now that we have developed coordinate systems for planes, we are able to discuss analytic representations of subsets of planes. We start with the line.

Symmetric Form. In Section 2-3 we developed Equation (1),

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}, \tag{1}$$

which is the analytic condition describing a point $P = (x,y)$ on the oblique line $\overleftrightarrow{P_0P_1}$, where $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$. (We note that the requirement that the line be oblique ensures that the denominator in each member is not zero.)

Since every point on the line may be described in this way,

$$\left\{(x,y) : \frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}\right\} = \overleftrightarrow{P_0P_1} .$$

We call Equation (1) a symmetric form of the equation of a line.

Example 1. A symmetric form of an equation of the line containing the points $(2,3)$ and $(4,-1)$ is

$$\frac{x - 2}{4 - 2} = \frac{y - 3}{-1 - 3} \quad \text{or} \quad \frac{x - 2}{2} = \frac{y - 3}{-4} .$$

Two-Point Form. If we reverse the order of the members of Equation (1) and multiply by $(y_1 - y_0)$, we obtain

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) . \tag{2}$$

We call Equation (2) a two-point form of the equation of a line.

Example 2. A two-point form for an equation of the line containing the points $(1,-2)$ and $(4,5)$ is

$$y + 2 = \frac{5 + 2}{4 - 1}(x - 1) \quad \text{or} \quad y + 2 = \frac{7}{3}(x - 1) .$$

We note that in Equation (2) the quotient of differences, or the difference quotient, $\frac{y_1 - y_0}{x_1 - x_0}$ is, by definition, the slope of the segment $\overline{P_0P_1}$. In your study of geometry you may have used similar triangles to prove that every segment of a given line has the same slope. We define the slope of a line to be the slope of all the segments on that line. We denote the slope of a segment or line by m.

The two-point form is not precisely equivalent to the symmetric form, since it is also defined when $y_0 = y_1$ or $y_1 - y_0 = 0$. In this case the line $\overleftrightarrow{P_0P_1}$ is parallel to the x-axis, has a slope of zero, and is represented by the equation $y - y_0 = 0$.

If $x_0 = x_1$ or $x_1 - x_0 = 0$, neither the symmetric form nor the two-point form as given in Equation (2) is defined. In this case an alternative two-point form

$$x - x_0 = \frac{x_1 - x_0}{y_1 - y_0}(y - y_0) \tag{3}$$

is defined. In this case the line $\overleftrightarrow{P_0P_1}$ has no slope, is parallel to the y-axis, and is represented by the equation $x - x_0 = 0$.

If $x_0 = x_1$ and $y_0 = y_1$, the points P_0 and P_1 are, of course, not distinct and no line is determined.

Example 3.

(a) The line containing the points (1,2) and (4,3) has slope $m = \frac{3 - 2}{4 - 1} = \frac{1}{3}$ and has as an equation in two-point form

$$y - 2 = \frac{3 - 2}{4 - 1}(x - 4) \quad \text{or} \quad y - 2 = \frac{1}{3}(x - 4) .$$

(b) The line containing the points (2,3) and (4,3) has slope $m = \frac{3 - 3}{4 - 2} = 0$ and has an equation in two-point form

$$y - 3 = \frac{3 - 3}{4 - 2}(x - 2) \quad \text{or} \quad y - 3 = 0 .$$

The line containing the points $(1,3)$ and $(1,5)$ has no slope since $\frac{5-3}{1-1} = \frac{2}{0}$ is not defined. However, it has an equation in an alternative two-point form:

$$x - 1 = \frac{1-1}{5-3}(y-3) \quad \text{or} \quad x - 1 = 0 .$$

Point-Slope Form. Since a line is determined by two distinct points, a line in a plane with a rectangular coordinate system is determined by the coordinates of two points on the line. If a line has slope, it is also determined by its slope and the coordinates of one of its points.

If a line has slope m and contains the point (x_0,y_0), we may replace the difference quotient in Equation (2) by m to obtain

(4) $$y - y_0 = m(x - x_0) .$$

We call Equation (4) a point-slope form of the equation of a line.

Example 4. A point-slope form of the line which contains the point $(5,-3)$ and has slope $\frac{2}{3}$ is

$$y + 3 = \frac{2}{3}(x - 5) .$$

Inclination. Occasionally we wish to describe a line, not by its slope, but by an angle related to the slope.

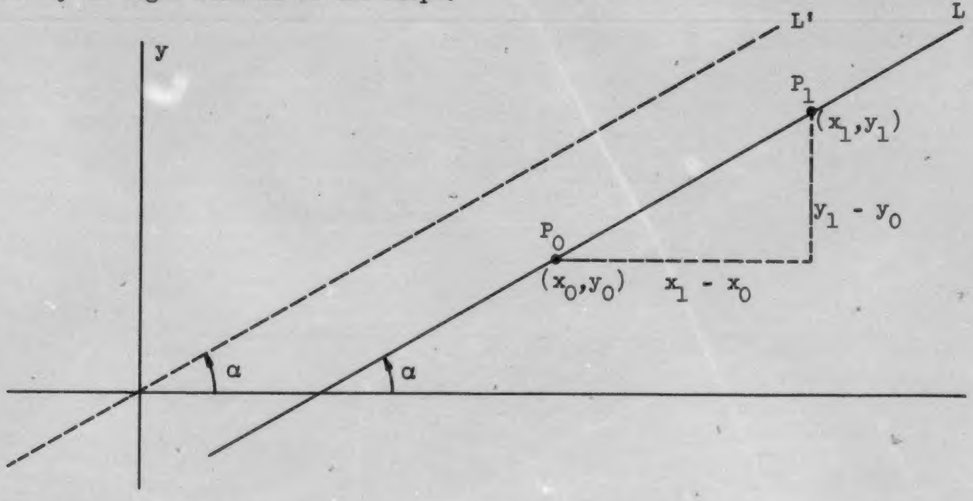


Figure 2-12

In Figure 2-12 the angle α is the angle of inclination of line L. The measure of angle α is the inclination of L. The angle α has the same measure as the corresponding angle measured in a counterclockwise direction from the positive side of the x-axis to the unique line L' which is parallel to L and contains the origin. (If L contains the origin, angle α corresponds to itself.)

We observe that if L is the x-axis or is parallel to the x-axis, its inclination is 0°. We also note that the slope of L is the tangent of angle α. If $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$, then for the line $\overleftrightarrow{P_0P_1}$

$$\tan \alpha = m = \frac{y_1 - y_0}{x_1 - x_0} .$$

For an angle α measured in degrees or radians, it is always the case that $0 \leq \alpha < 180^\circ$ or $0 \leq \alpha < \pi$, respectively.

Example 5.

(a) If the slope of a line is $\sqrt{3}$, then $\tan \alpha = \sqrt{3}$ and the inclination α of the line is 60° or $\frac{\pi}{3}$.

(b) For the line containing the points $(-1,4)$ and $(2,7)$

$$\tan \alpha = m = \frac{7 - 4}{2 + 1} = 1 \text{ and } \alpha = 45^\circ \text{ or } \frac{\pi}{4} .$$

Slope-Intercept Form. The x-intercepts of any graph are the abscissas of the points of the graph which are on the x-axis. The y-intercepts are the ordinates of the points of the graph on the y-axis.

A line has a unique y-intercept if and only if its slope is defined. If the slope is defined, the line is distinct from the y-axis and is not parallel to the y-axis. The line intersects the y-axis in a single point and therefore has a unique y-intercept. If the slope is not defined, the line either is the y-axis or is parallel to the y-axis. In either case the intersection of the line and the y-axis does not contain a unique point.

Since the lines with unique y-intercepts are those for which the slope is defined, they are the same lines which have point-slope forms. The point-slope form

$$y - y_0 = m(x - x_0)$$

is equivalent to

$$y = mx + (y_0 - mx_0) . \tag{5}$$

We observe that the y-intercept is the y-coordinate of the point whose x-coordinate is zero. If we let $x = 0$ in Equation (5), we find that the y-intercept is $y_0 - mx_0$. It is customary to denote the y-intercept by b. With this change Equation (5) becomes

$$y = mx + b , \tag{6}$$

which is called the slope-intercept form of the equation.

Example 6.

(a) The line with slope 3 and y-intercept -7 is represented by the equation $y = 3x - 7$.

(b) The line represented by the equation

$$\frac{y - 2}{3} = \frac{x + 4}{7} ,$$

which is equivalent to

$$y = \frac{3}{7}x + \frac{12}{7} + 2$$

or

$$y = \frac{3}{7}x + \frac{26}{7} ,$$

has slope $\frac{3}{7}$ and y-intercept $\frac{26}{7}$.

Intercept Form. A line has a unique x-intercept if and only if it does not have zero slope. The slope is zero if and only if the line either is the x-axis or is parallel to the x-axis. The line is not the x-axis and is not parallel to the x-axis if and only if it intersects the x-axis in a single point. In this case the x-intercept is unique.

It is customary to denote a unique x-intercept by a.

If the slope of a line is defined and is not zero, both intercepts are unique. Since the x-intercept is the x-coordinate of the point whose y-coordinate is zero, we let y be zero in Equation (6) and find that the x-intercept $a = \frac{-b}{m}$. If in addition $ab \neq 0$ (neither a nor b is zero), we may transform Equation (6)

$$y = mx + b$$

to obtain

$$-\frac{mx}{b} + \frac{y}{b} = 1$$

or

$$\frac{x}{-\frac{b}{m}} + \frac{y}{b} = 1 .$$

We substitute the value of the x-intercept to obtain

(7) $$\frac{x}{a} + \frac{y}{b} = 1 .$$

This is called the intercept form of the equation of a line.

Example 7. Find the intercept form of an equation for the line containing the points $(-1,4)$ and $(2,5)$.

Solution.

(a) The line has an equation in two-point form,

$$y - 4 = \frac{5 - 4}{2 + 1}(x + 1)$$

or

$$y - 4 = \frac{1}{3}(x + 1)$$

or

$$y = \frac{1}{3}x + \frac{13}{3} .$$

The y-intercept is $\frac{13}{3}$ and when $y = 0$, $x = -13$. Hence the x-intercept is -13 and the intercept form is

$$\frac{x}{-13} + \frac{y}{\frac{13}{3}} = 1 .$$

(b) If the intercepts are a and b , then the line contains the points $(a,0)$ and $(0,b)$. Since the slope is $\frac{5 - 4}{2 + 1} = \frac{1}{3}$, it must also be the case that

$$\frac{5 - 0}{2 - a} = \frac{1}{3} \quad \text{and} \quad \frac{5 - b}{2 - 0} = \frac{1}{3} ,$$

or

$$a = -13 \quad \text{and} \quad b = \frac{13}{3} .$$

Hence, the intercept form is

$$\frac{x}{-13} + \frac{y}{\frac{13}{3}} = 1 .$$

General Form. Each of the preceding forms of the equation of a line has certain advantages, not only because it is easy to write when certain facts about the line are known, but also because each clearly displays in its written form certain geometric properties of the line. However, none of these forms is defined for all lines.

The symmetric form

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}$$

is not defined for a line parallel to either axis, but if we transform the equation to

$$(y_1 - y_0)(x - x_0) = (x_1 - x_0)(y - y_0) , \text{ where } x_0 \neq x_1 \text{ or } y_0 \neq y_1$$

the new equation does describe any line in the plane. In order to simplify this equation, we collect all non-zero terms in one member of the equation and identify the coefficients of x and y and the constant term.

$$(y_1 - y_0)x - (x_1 - x_0)y - x_0(y_1 - y_0) + y_0(x_1 - x_0) = 0$$

is equivalent to

$$(y_1 - y_0)x + (x_0 - x_1)y + (x_1y_0 - x_0y_1) = 0 .$$

We let

$$a = y_1 - y_0 , \; b = x_0 - x_1 , \text{ and } c = x_1y_0 - x_0y_1 ,$$

and write

(8) $ax + by + c = 0$, where $a^2 + b^2 \neq 0$ (that is, $a \neq 0$ or $b \neq 0$) .

Equation (8) is called a general form of the equation of a line. It is also called the general linear equation in x and y .

Example 8. Write the equations

(a) $3x + 4y - 8 = 0$ and

(b) $ax + by + c = 0$, where $abc \neq 0$, (that is, $a \neq 0$, $b \neq 0$, and $c \neq 0$) in intercept and slope-intercept form.

Solution.

(a)
$$3x + 4y - 8 = 0$$

is equivalent to

$$\frac{3x}{8} + \frac{y}{2} = 1$$

or

$$\frac{x}{\frac{8}{3}} + \frac{y}{2} = 1 ,$$

which is in the intercept form.

The original equation is also equivalent to

$$4y = -3x + 8$$

and

$$y = -\frac{3}{4}x + 2 ,$$

which is in the slope-intercept form.

(b)
$$ax + by + c = 0 \text{ , where } abc \neq 0 ,$$

is equivalent to

$$\frac{ax}{-c} + \frac{by}{-c} = 1 \text{ , where } abc \neq 0 ,$$

and

$$\frac{x}{-\frac{c}{a}} + \frac{y}{-\frac{c}{b}} = 1 \text{ , where } abc \neq 0 ,$$

which is in the intercept form.

$$ax + by + c = 0 \text{ , where } abc \neq 0 \text{ ,}$$

is equivalent to

$$by = -ax - c \text{ , where } abc \neq 0 \text{ ,}$$

and

$$y = -\frac{a}{b}x - \frac{c}{b} \text{ , where } abc \neq 0 \text{ ,}$$

which is in the slope-intercept form.

From Example 8(b) we observe that when an equation of a line is expressed in general form, the x- and y-intercepts are $-\frac{c}{a}$ and $-\frac{c}{b}$ respectively if they exist and the slope of the line is $-\frac{a}{b}$ if it is defined.

The great advantage of the general form is that it can be written for any line. The only shortcoming is that the geometric properties of the line are less clearly revealed by this form.

Exercises 2-5

1. Use Equation (4) to find an equation of a line containing $(2,-3)$ and having slope 2 . Put the equation in general form. If the line contains the points $(p,7)$ and $(5,q)$, find p and q .
2. Find an equation of a line with slope $-\frac{2}{3}$ and passing through $(-3,5)$. If this line contains the points $(p,7)$ and $(5,q)$, find p and q .
3. Find an equation of a line containing the point $(0,b)$ and having slope 3 . If the line contains the points $(p,7)$ and $(5,q)$, find p and q .
4. Find an equation of a line containing the point $(4,5)$ and having the same slope as the line $2x - 3y = 600$. Describe the relative position of these two lines.
5. Write an equation of a line having slope k and containing the point $(a,0)$. What are the coordinates of the point where the line crosses the y-axis?

6. Write an equation representing all lines containing the origin. Are you sure every line is represented by your equation? Write the equation of the one of these lines that contains the point $(-3,5)$.

7. The coordinates of A and B are $(3,5)$ and $(-5,3)$. Segments $\overline{OA}$ and $\overline{OB}$ form a right angle at the origin. Determine the slope of each segment and try to arrive at a general conclusion that you can prove.

8. Choose $(-8,8)$ as (x_0,y_0) and write the equation $3x + 4y - 8 = 0$ in symmetric form.

9. Write an equation of the line containing the points $(-4,8)$ and $(2,3)$. Exhibit the result in all seven forms so far discussed. What is the slope? what are the intercepts?

10. Write the equation $ax + by + c = 0$ in the slope-intercept form. What is the geometric interpretation of $ax + by + c = 0$,

    (a) when $b = 0$, $ac \neq 0$?

    (b) when $a = 0$, $bc \neq 0$?

    (c) when $c = 0$, $ab \neq 0$?

11. Find an equation of a line satisfying the following conditions:

    (a) Containing the point $(3,-2)$ and having y-intercept 5 .

    (b) Containing the point $(3,-2)$ and having x-intercept 5 .

    (c) Containing the midpoint of $\overline{AB}$ where $A = (-7,2)$, $B = (3,4)$. and with the same slope as the line OA .

    (d) Containing the point $(2,-4)$ and with inclination 135° .

    (e) Containing the point $(-1,-3)$ and with inclination 30° .

12. In triangle ABC , $A = (1,-2)$, $B = (3,2)$, $C = (0,4)$. Find an equation of each of the following lines:

    (a) $\overleftrightarrow{AB}$.

    (b) The median from A .

    (c) The line joining the midpoints of $\overline{AC}$ and $\overline{BC}$.

13. Find an equation of a line containing the point $P = (5,8)$ which forms with the coordinate axes a triangle with area 10 square units.

Review Exercises--Section 2-1 through Section 2-5

In Exercises 1-4 find the graph of the sets described on a line with a linear coordinate system.

1. $\{x : 1 < x \leq 2\}$.
2. $\{x : (x - 1)(x + 2) = 0\}$.
3. $\{x : |x| < 3\}$.
4. $\{x : |x - 4| \geq 2\}$.

In Exercises 5 to 9 graph and describe the geometric representation in one-space and 2-space.

5. $\{x : x + 4 = 0\}$.
6. $\{x : |x| + 4 = 0\}$.
7. $\{x : 2 < x < 6\}$.
8. $\{x : 2 \leq |x|\}$.
9. $\{x : |x| \leq 6\}$.
10. Find the midpoints and trisection points of
    (a) $\overline{AB} = \{x : -1 \leq x \leq 2\}$.
    (b) $\overline{BC} = \{x : |x + 2| \leq 3\}$.
    (c) $\overline{CD} = \{x : c \leq x \leq d \text{ , } (c + 2)(d - 3) = 0\}$.
11. Find a polar representation for the points whose rectangular coordinates are:
    (a) $(1,\sqrt{3})$.
    (b) $(-\sqrt{2},\sqrt{2})$.
    (c) $(3,-4)$.
    (d) $(-2,-3)$.
    (e) $(1,0)$.
    (f) $(0,1)$.
12. Find the rectangular representation for the points whose polar coordinates are:
    (a) $(4,45^\circ)$.
    (b) $(3,\frac{2\pi}{3})$.
    (c) $(-2,\frac{5\pi}{4})$.
    (d) $(6,\frac{9\pi}{4})$.
    (e) $(5,-135^\circ)$.
    (f) $(-3,-750^\circ)$.

In each exercise from 13 to 18 write an equation of a line which satisfies the given conditions.

13. Contains $(-2,5)$; $m = -\frac{3}{4}$.

14. Contains $(-3,2)$, $(8,10)$.

15. Contains $(-4,-5)$, $(-6,-10)$.

16. Contains $(4,5)$; $\alpha = 120^\circ$.

17. Horizontal; y-intercept 6 .

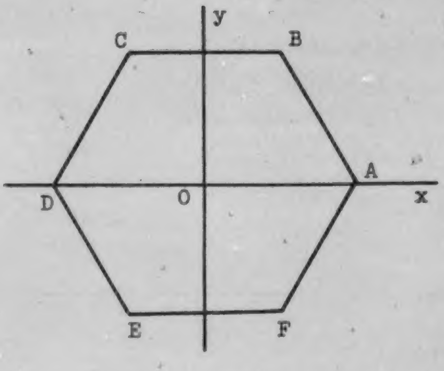
18. Vertical; x-intercept 4 .

Exercises 19 - 25 refer to the figure at the right, which represents a regular hexagon with sides of length 6 . The coordinates of the vertices are:

$A = (6,0)$; $B = (3,3\sqrt{3})$;

$C = (-3,3\sqrt{3})$; $D = (-6,0)$;

$E = (-3,-3\sqrt{3})$; $F = (3,-3\sqrt{3})$.



19. Write equations of the lines determined by each of the six sides in slope-intercept form.

20. Write equations of the lines determined by each of the six sides in general form.

21. Write equations of the lines determined by each of the six sides in symmetric form.

22. Find the slopes of $\overleftrightarrow{AC}$, $\overleftrightarrow{BD}$, $\overleftrightarrow{AE}$, and $\overleftrightarrow{DF}$.

23. Find the coordinates of the two trisection points of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{FA}$.

24. Find coordinates of the points P , Q , and R , where

(a) P is on $\overleftrightarrow{AB}$ and $\frac{d(A,P)}{d(A,B)} = \frac{2}{3}$ (two answers).

(b) Q is on $\overleftrightarrow{BC}$ and $\frac{d(B,Q)}{d(Q,C)} = \frac{3}{4}$ (two answers).

(c) R is on $\overleftrightarrow{CD}$ and $\frac{d(C,R)}{d(R,D)} = \frac{4}{5}$ (two answers).

25. Find the inclination, to the nearest degree, of $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AE}$, and $\overleftrightarrow{AF}$.

26. Summarize the different forms of the equation of a line in a table, listing for each form its particular advantages and disadvantages.

Which form, or forms, of equations for a line would you use to answer each of the following questions in the most efficient way? Be prepared to explain your answer.

(a) Is the point $(3,7)$ on the line?
(b) Does the line intersect the x-axis? If so, where?
(c) Does the line contain the origin?
(d) What is the slope of the line?
(e) Find the ordinate of the point where the abscissa is 5 .
(f) Find the point on the line where the two coordinates are equal.
(g) If the point $(3,3-k)$ is on the line, find k .
(h) Suppose the point P is on the line; find the points R and S on the line which are 5 units from P .

Graph the relations of Exercises 27 to 32.

27. $\{(x,y) : |x| + |y| - 10 = 0\}$.

28. $\{(x,y) : |x| + |y| = 0\}$.

29. $\{(x,y) : x - y < 1\}$.

30. $\{(x,y) : x - y \leq 1\}$.

31. $\{(x,y) : x - y \leq 1\} \cap \{(x,y) : x + y < 1\}$.

32. $R_1 = \{(x,y) : |x| < 4\}$, $R_2 = \{(x,y) : |y| < 4\}$, $R_3 = R_1 \cap R_2$.

33. Discuss Exercise 32 if $<$ is changed to $\leq$. What geometric interpretation can you give for $R_1 \cup R_2$?

34. Two thermometers in common use are the Fahrenheit and Centigrade. The freezing point for water is $32^{\circ}F$ and $0^{\circ}C$; the boiling point for water is $212^{\circ}F$ and $100^{\circ}C$. Derive a formula for expressing temperature on one scale in terms of the other. Find the temperature reading which gives the same number on both scales.

35. Graph the following relations:

(a) $R_1 = \{(x,y) : 2x + 3y - 6 = 0\}$.

(b) $R_2 = \{(x,y) : 7x + y - 2 = 0\}$.

(c) $R_3 = \{(x,y) : 5x - 2y - 15 = 0\}$.

(d) $R_4 = \{(x,y) : 2x + 3y \leq 6\}$.

(e) $R_5 = \{(x,y) : 7x + y \geq 2\}$.

(f) $R_6 = \{(x,y) : 5x - 2y \leq 15\}$.

(g) $R_4 \cap R_5 \cap R_6$.

Challenge Exercises

Note: The symbol $[x]$ is used to represent the first integer $\leq x$, or stated in another way, $[x]$ means the greatest integer not greater than x . For instance, if $0 < x < 1$, $[x] = 0$; if $x = 2$, $[x] = 2$; if $-1 < x < 0$, $[x] = -1$.

Graph the relations.

1. (a) $R_1 = \{(x,y) : [x] = x\}$.

   (b) $R_2 = \{(x,y) : [y] = y\}$.

   (c) $R_3 = \{(x,y) : [x] = x\} \cap \{(x,y) : [y] = y\}$.

   (d) $R_4 = \{(x,y) : [x] = x\} \cup \{(x,y) : [y] = y\}$.

   (e) $R_5 = \{(x,y) : [x] = [y]\}$.

   (f) $R_6 = \{(x,y) : [x] = [y + k]\}$.

   (g) $R_7 = \{(x,y) : [x] = [-y]\}$.

   (h) $R_8 = \{(x,y) : [x] = -[y]\}$.

2. Graph $r = \theta$.

3. Graph $r^2 = \theta$.

4. When we introduced a system of rectangular coordinates into a plane, we used on each axis linear coordinate systems in the same units. Then if $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ are any two points in the plane,

$$d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$

Suppose instead that on the x- and y-axes we use linear coordinate systems for which the units are in the ratio r to s respectively, where $r \neq s$.

(a) Find a formula for $d(P_1,P_2)$ in the units of the x-axis.

(b) Find a formula for $d(P_1,P_2)$ in the units of the y-axis.

(c) Let P, Q, R, and S be four points in the plane, with coordinates (p_1,p_2), (q_1,q_2), (r_1,r_2), and (s_1,s_2) respectively. Under what conditions is $\overline{PQ} \cong \overline{RS}$ and

$$\sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} = \sqrt{(r_1 - s_1)^2 + (r_2 - s_2)^2} \ ?$$

5. Find the graph of $S = \{(x,y) : (4x + 3y - 5)^2 = 0\}$. Can you find a simpler analytic representation for the graph?

6. What is the graph of $T = \{(x,y) : (ax + by + c)^k = 0$, where $a^2 + b^2 \neq 0$ and k is a positive integer$\}$? Can you find a simpler analytic representation for the graph?

7. Find the intersection of $L_1 = \{(x,y) : 3x + 2y - 1 = 0\}$ and $L_2 = \{(x,y) : 2x - 3y + 2 = 0\}$.

8. Find the graph of $U = \{(x,y) : (3x + 2y - 1)(2x - 3y + 2) = 0\}$.

9. Find the graph of $V = \{(x,y) : (x + y)(x - y) = 0\}$.

10. Find the graph of $W = \{(x,y) : xy = 0\}$.

11. Assume that $L_0 = \{(x,y) : a_0x + b_0y + c_0 = 0 , a_0^2 + b_0^2 \neq 0\}$ and $L_1 = \{(x,y) : a_1x + b_1y + c_1 = 0 , a_1^2 + b_1^2 \neq 0\}$ have a unique point (x_1,y_1) in common. What can you say about x_1 and y_1 if a_0, a_1, b_0, b_1, c_0, and c_1 are

    (a) integral?

    (b) rational?

    (c) real?

    (d) complex?

12. What can you say about the graph of

    (a) $R = \{(x,y) : (3x - 2y + 2) + k(x + y + 1) = 0$, where k is constant$\}$?

    (b) $S = \{(x,y) : (x + y + 1) + k(3x - 2y + 2) = 0$, where k is constant$\}$?

    (c) $T = \{(x,y) : m(3x - 2y + 2) + n(x + y + 1) = 0$, where $m^2 + n^2 \neq 0$, and m and n are contant$\}$?

13. What can you say about the graph of

(a) $U = \{(x,y) : (3x - 2y + 2) + t(x + y + 1) = 0$, where t is a real variable$\}$?

(b) $V = \{(x,y) : (x + y + 1) + t(3x - 2y + 2) = 0$, where t is a real variable$\}$?

(c) $W = \{(x,y) : s(3x - 2y + 2) + t(x + y + 1) = 0$, where $s^2 + t^2 \neq 0$, and s and t are real variables$\}$?

14. Assume that the linear equations $a_0x + b_0y + c_0 = 0$, where $a_0^2 + b_0^2 \neq 0$, and $a_1x + b_1y + c_1 = 0$, where $a_1^2 + b_1^2 \neq 0$, are not equivalent. What can you say about the graph of

(a) $R = \{(x,y) : (a_0x + b_0y + c_0) + k(a_1x + b_1y + c_1) = 0$, where k is constant$\}$?

(b) $S = \{(x,y) : (a_1x + b_1y + c_1) + k(a_0x + b_0y + c_0) = 0$, where k is constant$\}$?

(c) $T = \{(x,y) : (a_0x + b_0y + c_0) + t(a_1x + b_1y + c_1) = 0$, where t is real$\}$?

(d) $U = \{(x,y) : (a_1x + b_1y + c_1) + t(a_0x + b_0y + c_0) = 0$, where t is real$\}$?

(e) $V = \{(x,y) : m(a_0x + b_0y + c_0) + n(a_1x + b_1y + c_1) = 0$, where $m^2 + n^2 \neq 0$, and m and n are constant$\}$?

(f) $W = \{(x,y) : s(a_0x + b_0y + c_0) + t(a_1x + b_1y + c_1) = 0$, where $s^2 + t^2 \neq 0$, and s and t are real variables$\}$?

15. What is the graph of

(a) $S = \{(x,y) : 0 = 1\}$?

(b) $T = \{(x,y) : 1 = 1\}$?

2-6. Direction on a Line.

Although there are two senses of direction implicit in our intuitive notion of a line, neither one is dominant or primary. When we represent a line analytically, we may suggest a specific sense of direction for the line. When we undertake a geometric description of the line in terms of an associated angle, we suggest a sense of direction for the line if a side of the angle is contained in the line.

In this section we shall introduce some of the analytic ideas and terms which may be used once a sense of direction has been assumed for a line. We shall also consider the geometric interpretation of the ideas.

When we speak of the line segment from P_0 to P_1, we suggest a sense of direction on the line. If $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$, the numbers $\ell = x_1 - x_0$ and $m = y_1 - y_0$ also suggest this sense of direction.

The numbers ℓ and m are called direction numbers of L. For the ordered pair of direction numbers we use the symbol (ℓ,m). Since this symbol is also used for a point, care must be exercised to avoid ambiguity. Clearly a line has infinitely many pairs of direction numbers, since there are infinitely many pairs of points P_0 and P_1 which determine it. However, all the pairs for a given line L are related in a very simple way. If L has a slope and (ℓ,m) and (ℓ',m') are two pairs of direction numbers for L, then $\frac{m}{\ell} = \frac{m'}{\ell'}$ and there is a number $c \neq 0$ such that $\ell' = c\ell$ and $m' = cm$. If L has no slope, there is still such a number c, though the argument above does not prove it. If two lines are parallel, a similar argument shows that any two pairs of direction numbers for the two are related in the same way. Thus it is natural to make the following definition:

> DEFINITION. The pair (ℓ,m) of direction numbers is said to be equivalent to the pair (ℓ',m') if and only if there is a number $c \neq 0$ such that $\ell' = c\ell$, $m' = cm$.

The preceding discussion can now be summarized in the following statement.

> Two distinct lines in a plane are parallel if and only if any pair of direction numbers for one is equivalent to any pair for the other.

A pair (ℓ,m) of direction numbers for a line L may be said to determine a direction on the line in the following sense. Let $P_0 = (x_0,y_0)$ be a fixed point of L and $P = (x,y)$ any other point of L. Then $x - x_0 = c\ell$ and $y - y_0 = cm$, or

$$x = x_0 + c\ell ,$$
$$y = y_0 + cm , \quad \text{where} \quad c \neq 0 .$$

The point P_0 separates L into two sets of points; the points on one side of P_0 are given by positive values of c. P_0 and the points of L given by positive values of c form a ray, which we call the positive ray (on L) with endpoint P_0. If $P_1 = (x_1,y_1)$ is another point of L, then P_1 and the points $P = (x,y)$ given by

$$x = x_1 + c\ell ,$$
$$y = y_1 + cm , \quad \text{where} \quad c > 0 ,$$

form another positive ray on L. The intersection (set of common points) of the positive rays with endpoints P_0 and P_1 is one of those two rays. Intuitively speaking, all the positive rays point in the same direction on L. The pair $(c\ell,cm)$ of direction numbers determines the same direction on L as (ℓ,m) if and only if $c > 0$.

If (ℓ,m) is a pair of direction numbers for L, the equivalent pair

$$(\lambda,\mu) = \left(\frac{\ell}{\sqrt{\ell^2 + m^2}}, \frac{m}{\sqrt{\ell^2 + m^2}}\right)$$

is of particular importance. Such a pair is sometimes called a normalized pair. You should observe that $\lambda^2 + \mu^2 = 1$.

Let L be a line in a plane with a rectangular coordinate system and let L' be the line parallel to L which passes through the origin. (If L contains the origin, L' = L.) Then L and L' have the same pair of direction numbers (ℓ,m). Figure 2-13a shows the situation if $\ell > 0$ and $m > 0$, Figure 2-13b if $\ell > 0$ and $m < 0$, Figure 2-13c if $\ell < 0$ and $m < 0$, and Figure 2-13d if $\ell < 0$ and $m > 0$.

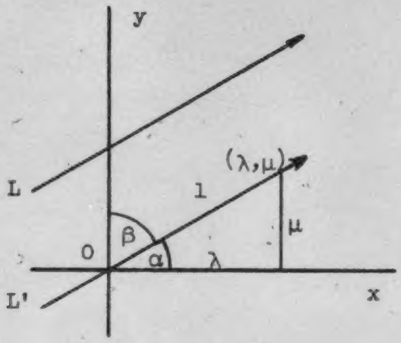


Figure 2-13a

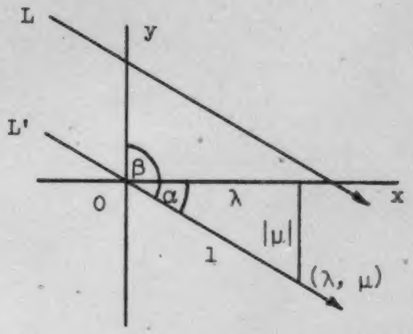


Figure 2-13b

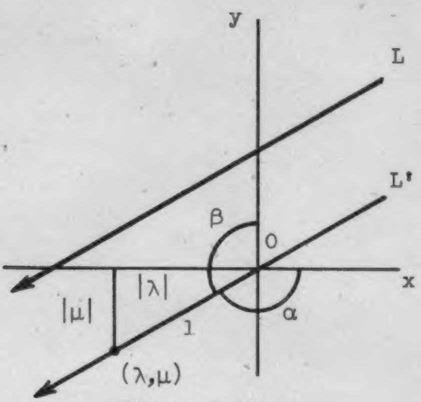


Figure 2-13c

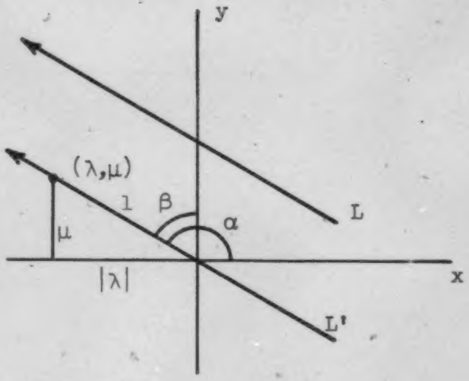


Figure 2-13d

The arrowheads show the positive directions on L and L' . The angles α and β are called the direction angles of the line L with the positive direction determined by the pair (ℓ,m) of direction numbers. $\angle\alpha$ is the angle formed by the positive ray on L' with the origin as endpoint, and the positive half of the x-axis. $\angle\beta$ is the angle formed by the positive ray on L' with the origin as endpoint, and the positive half of the y-axis. We note that the direction angles are geometric angles, with the single exception that their sides may be collinear. Hence, $0 \leq \alpha \leq 180^\circ$ and $0 \leq \beta \leq 180^\circ$.

If $c > 0$, each equivalent pair $(c\ell,cm)$ of direction numbers for L is also the pair of coordinates for a point on L' . The point with (λ,μ) , the normalized pair, as coordinates has been indicated in each case of Figure 2-13. Consideration of these cases reveals that, since $\lambda^2 + \mu^2 = 1$,

$\cos\alpha = \lambda$, and $\cos\beta = \mu$. The cosines of direction angles of a line L are called direction cosines for the line.

The direction numbers, angles, and cosines of a ray R are defined to be the direction numbers, angles, and cosines, respectively, of the line containing R with positive direction determined by R .

Example 1. What are the pairs of direction numbers for the line determined by the points $P_0 = (-2,7)$ and $P_1 = (6,-2)$?

Solution. One pair is $\big(-2 - 6, 7 - (-2)\big)$, or $(-8,9)$, but any equivalent pair $(-8c,9c)$, where $c \neq 0$, will do. Since any pair (ℓ,m) must be such that $\frac{\ell}{m} = \frac{-8}{9}$ or $9\ell + 8m = 0$, we may write this as

$\{(\ell,m) : 9\ell + 8m = 0 , \ell^2 + m^2 \neq 0\}$.

Example 2.

(a) What are the direction cosines and the measures of the direction angles for the line L with the positive direction determined by the pair $(1,1)$ of direction numbers?

(b) What are the direction cosines and angles for the same line L , but with the positive direction determined by the equivalent pair $(-1,-1)$?

Solution.

(a) $\cos\alpha = \lambda = \dfrac{\ell}{\sqrt{\ell^2 + m^2}}$ and $\cos\beta = \mu = \dfrac{m}{\sqrt{\ell^2 + m^2}}$.

Therefore, $\cos\alpha = \dfrac{1}{\sqrt{2}}$, $\cos\beta = \dfrac{1}{\sqrt{2}}$ and $\alpha = \beta = 45^\circ$.

(b) In this case, $\cos\alpha = \dfrac{-1}{\sqrt{2}}$, $\cos\beta = \dfrac{-1}{\sqrt{2}}$ and $\alpha = \beta = 135^\circ$.

Example 3. Find the direction angles and direction cosines of the line through $(1,2)$ with positive direction determined by the pair $(-\sqrt{3},1)$ of direction numbers. Do the same when the positive direction is determined by the pair $(\sqrt{3},-1)$.

Solution. In the first case, $\lambda = -\frac{\sqrt{3}}{2}$ and $\mu = \frac{1}{2}$. Since by definition $0 \le \alpha \le 180^\circ$ and $0 \le \beta \le 180^\circ$, and since $\cos \alpha = \lambda$ and $\cos \beta = \mu$, we see that $\alpha = 150^\circ$, $\beta = 60^\circ$. If we consider the other direction on L, we have $\cos \alpha = \frac{\sqrt{3}}{2}$, $\cos \beta = -\frac{1}{2}$. Hence $\alpha = 30^\circ$, $\beta = 120^\circ$.

Examples 2 and 3 suggest a careful distinction to be made. A line has unsensed direction, or perhaps it would be better to say that two opposite senses of direction are implied for a given line, but neither one is dominant. Some of the pairs of direction numbers for a line imply each sense, but if we select a single pair, we select a single sense of direction as well. Direction angles and direction cosines are defined only for a line with a specified sense of direction. We shall call such a line a directed line. The sense of direction may be specified by the context, such as the choice of a single pair of direction numbers for the line.

In Figure 2-14 we observe that either $\angle\alpha$ and $\angle\beta$ or $\angle\alpha'$ and $\angle\beta'$ might be direction angles for line L. Since $\alpha + \alpha' = 180^\circ$ and $\beta + \beta' = 180^\circ$, we note that $\cos \alpha' = -\cos \alpha$ and $\cos \beta = -\cos \beta$. Thus, if the normalized pair (λ, μ) of direction numbers are direction cosines for a directed line, $(-\lambda, -\mu)$ are the pair of direction cosines for the same line with opposite direction; if $\angle\alpha$ and $\angle\beta$ are direction angles for a directed line, their supplements are direction angles for the same line with opposite direction.

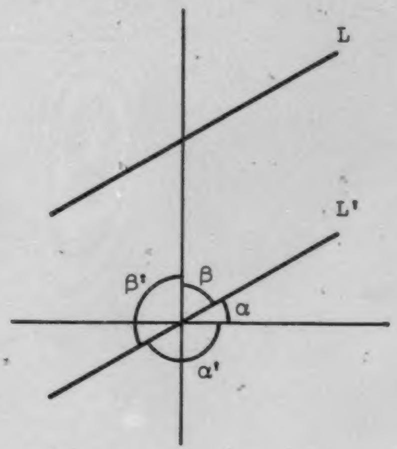


Figure 2-14

Example 4. Find direction numbers, cosines, and angles for the lines

(a) $\{(x,y) : 3x - 4y - 5 = 0\}$ and

(b) $\{(x,y) : ax + by + c = 0\}$, $b \neq 0\}$.

Solution.

(a) We observe that if a nonvertical line has a pair (ℓ,m) of direction numbers and an equation in general form, $ax + by + c = 0$, then the slope of the line is given by both $\frac{m}{\ell}$ and $-\frac{a}{b}$. Therefore

$$\frac{m}{\ell} = -\frac{a}{b}\text{, where } \ell b \neq 0 .$$

Since $3x - 4y - 5 = 0$ is in general form, the slope of the line is $\frac{3}{4}$, $(4,3)$ is a pair of direction numbers, and any other pair $(4c,3c)$, where $c \neq 0$, is an equivalent pair of direction numbers. The normalized pair (λ,μ) of direction numbers, or direction cosines $\cos\alpha$ and $\cos\beta$, is either

$$\left(\frac{4}{\sqrt{4^2+3^2}}, \frac{3}{\sqrt{4^2+3^2}}\right) = \left(\frac{4}{5}, \frac{3}{5}\right) \text{ or } \left(-\frac{4}{5}, -\frac{3}{5}\right),$$

depending on which sense of direction is adopted for the line. We use tables of trigonometric functions to discover that the measures α and β of the corresponding direction angles are (approximately) 37° and 53°, or 143° and 127° respectively.

(b) For the general form of an equation of a line $ax + by + c = 0$, where $b \neq 0$, the slope is $-\frac{a}{b}$. Thus, $(-b,a)$, $(b,-a)$, and, in general, $(-bk,ak)$, where $k \neq 0$, are pairs of direction numbers. The normalized pair, or pair of direction cosines, is

$$\left(\frac{-b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\right) \text{ or } \left(\frac{b}{\sqrt{a^2+b^2}}, \frac{-a}{\sqrt{a^2+b^2}}\right),$$

depending on the sense of direction. Once the direction cosines are found, the direction angles are uniquely determined, since by definition $0 \leq \alpha \leq 180^\circ$ and $0 \leq \beta \leq 180^\circ$.

Example 5. Consider the line $L = \{(x,y) : \frac{x}{a} + \frac{y}{b} = 1 \text{ , } ab \neq 0\}$. Let O be the origin; let A and B be the points of L on the x- and y-axes respectively.

(a) Write an equation of L in general form.

(b) Find the length of the altitude $\overline{OC}$ on the hypotenuse of right triangle AOB .

(c) Find the direction cosines of $\overrightarrow{OC}$.

(d) How are the coefficients in the answer to Part (a) related to the results of Parts (c) and (b)?

Solution.

(a) $\frac{x}{a} + \frac{y}{b} = 1$ is equivalent to $bx + ay - ab = 0$, which is in general form.

(b) The area of $\triangle AOB$ is equal both to $\frac{1}{2}|ab|$ and to $\frac{1}{2}\sqrt{a^2 + b^2} \cdot d(O,C)$; hence, $\frac{1}{2}|ab| = \frac{1}{2}\sqrt{a^2 + b^2} \cdot d(O,C)$.

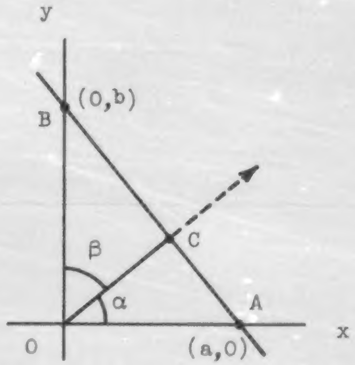
Therefore, the length of $\overline{OC} = d(O,C) = \dfrac{|ab|}{\sqrt{a^2 + b^2}}$.

(c) $\cos \alpha = \cos \angle ABO = \dfrac{b}{\sqrt{a^2 + b^2}}$. (Why?)

$\cos \beta = \cos \angle BAO = \dfrac{a}{\sqrt{a^2 + b^2}}$. (Why?)

(d) Lastly, we note that the results of Parts (c) , and (b) apart from a possible difference in sign, are proportional to the coefficients in the equation obtained in Part (a). The constant of proportionality is

$$\frac{1}{\sqrt{a^2 + b^2}} \left(\text{or } \frac{-1}{\sqrt{a^2 + b^2}}\right).$$



Exercises 2-6

1. Find pairs of direction numbers for the line through each pair of points given below. Use both possible orders.

   (a) $(5,-1)$, $(2,3)$
   (b) $(0,0)$, $(4,1)$
   (c) $(2,-3)$, $(2,3)$
   (d) $(-1,4)$, $(-6,4)$
   (e) $(1,1)$, $(2,2)$
   (f) $(-1,-1)$, $(1,1)$
   (g) $(1,0)$, $(0,1)$
   (h) $(2,-2)$, $(-2,2)$

2. Find the normalized pairs of direction numbers for the lines in Exercise 1.

3. Find the direction angles of the lines in Exercises 1 and 2.

4. Given the pairs $(3,-4)$, $(2,0)$, $(0,-3)$, $(-1,2)$, and $(-2,1)$ of direction number,

   (a) find the slope of a line with each pair as a pair of direction numbers
   (b) find a pair equivalent to each pair, and find the corresponding direction angles
   (c) draw the line through the origin with each pair as its direction numbers, and indicate the positive direction on each line determined by the pair (Do not draw too many on one sketch.)
   (d) indicate on your sketches the direction angles of each directed line.

5. Let $P_0 = (x_0,y_0)$, $P_1 = (x_0,y_1)$, and $P_2 = (x_0,y_2)$ be any three distinct points on a line parallel to the y-axis in a plane with a rectangular coordinate system. Show that the pair of direction numbers determined by P_0 and P_1 and the pair of direction numbers determined by P_0 and P_2 are equivalent.

6. Let α and β be the direction angles of the line L with positive direction determined by the pair (ℓ,m) of direction numbers, α' and β' the direction angles of L with positive direction determined by the pair $(-\ell,-m)$ of direction numbers. Prove that α and α' are supplementary, and that β and β' are supplementary.

7. Assume that in each part of Figure 2-13 a polar coordinate system has also been introduced in the usual way. Let ω denote the measure of a polar angle which contains the positive ray of L' with endpoint at the origin.

   (a) Show that in each case $\sin \omega = \cos \beta$.
   (b) Show that $\sin \omega = \cos \beta$ for any positive ray lying on an axis.

8. Find pairs of direction numbers, direction cosines, and direction angles for the lines L, M, and N, where

(a) $L = \{(x,y) : x - 2y + 7 = 0\}$

(b) $M = \{(x,y) : y = -\frac{1}{2}x + 7\}$

(c) $N = \{(x,y) : \frac{x}{6} - \frac{y}{5} = 1\}$.

2-7. The Angle Between Two Lines; Parallel and Perpendicular Lines.

We have developed various forms of an equation of a line. Here we shall use equations to answer a question about the lines they represent: What angle is formed by two lines? In particular, are two lines perpendicular or parallel?

We observed that the slope of lines parallel to the x-axis is zero, and that lines parallel to the y-axis have no slope. Because of the customary orientation of the axes we usually refer to lines parallel to the x-axis as *horizontal lines* and to lines parallel to the y-axis as *vertical lines*.

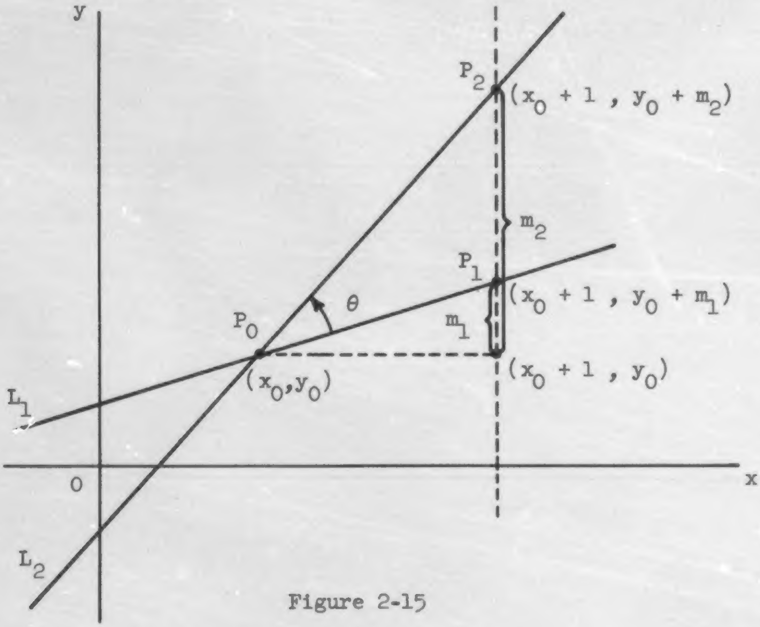


Figure 2-15

In Figure 2-15 we indicate two nonvertical lines L_1 and L_2, intersecting at the point $P_0 = (x_0,y_0)$. The vertical line represented by the equation $x = x_0 + 1$ will intersect these lines at P_1 and P_2 respectively. If we represent the slopes of L_1 and L_2 by m_1 and m_2 respectively, the coordinates of P_1 and P_2 will be $(x_0 + 1 , y_0 + m_1)$ and $(x_0 + 1, y_0 + m_2)$ respectively. If in triangle $P_0P_1P_2$ we apply the distance formula and the Law of Cosines in terms of $\angle P_1P_0P_2 = \angle\theta$, we obtain

$$\left(d(P_1,P_2)\right)^2 = \left(d(P_0,P_1)\right)^2 + \left(d(P_0,P_2)\right)^2 - 2d(P_0,P_1)d(P_0,P_2)\cos\theta ,$$

or

$$(m_2 - m_1)^2 = 1 + m_1^2 + 1 + m_2^2 - 2\sqrt{1 + m_1^2}\cdot\sqrt{1 + m_2^2}\cos\theta .$$

This is equivalent to

$$-2m_1m_2 = 2 - 2\sqrt{1 + m_1^2}\cdot\sqrt{1 + m_2^2}\cos\theta ,$$

or

$$\cos\theta = \frac{1 + m_1m_2}{\sqrt{1 + m_1^2}\sqrt{1 + m_2^2}} . \tag{1}$$

Example 1. Find the measures of the angles of intersection between the lines represented by the equations $y = \frac{1}{3}x + 1$ and $y = 2x + 1$.

Solution. Since the equations are in slope-intercept form, we perceive immediately that the slopes of the lines are $\frac{1}{3}$ and 2. We substitute these values in Equation (1) to obtain

$$\cos\theta = \frac{1 + (\frac{1}{3})(2)}{\sqrt{1 + (\frac{1}{3})^2}\cdot\sqrt{1 + 2^2}} = \frac{\frac{5}{3}}{\frac{\sqrt{10}}{9}\cdot\sqrt{5}} = \frac{\frac{5}{3}}{\frac{5}{3}\sqrt{2}} = \frac{1}{\sqrt{2}} .$$

Thus $\theta = 45^\circ$, and the other three angles of intersection will have measures of 45°, 135°, and 135°.

In your previous courses you discovered that two nonvertical lines are parallel or the same if and only if they have the same slope. Clearly all vertical lines are parallel. You also discovered that two nonvertical lines are perpendicular if and only if the product of their slopes is -1 . It should be clear that a vertical line is perpendicular to a second line if and only if the second line is horizontal.

In Equation (1) we note that the lines are perpendicular if and only if

$$\cos \theta = 0, \text{ or } m_1 m_2 = -1 . \tag{2}$$

Example 2. Find an equation for the line L which contains the point $P = (4,3)$ and which is perpendicular to the line represented by the equation $2x + 3y + 7 = 0$.

Solution 1. In the previous section we observed that the slope of a line represented by an equation with general form $ax + by + c = 0$, $(b \neq 0)$, is $-\frac{a}{b}$. Thus the line above has slope $-\frac{2}{3}$. If L is perpendicular to the given line, its slope m must be such that

$$-\frac{2}{3}m = -1 , \text{ or } m = \frac{3}{2} .$$

Since L contains $P = (4,3)$, it has the equation in point-slope form,

$$(y - 3) = \frac{3}{2}(x - 4) .$$

This is equivalent to

$$\frac{3}{2}x - y - 3 = 0 ,$$

or

$$3x - 2y - 6 = 0 .$$

Solution 2. We might have developed a more general equation for a line L which contains $P_0 = (x_0,y_0)$, and which is perpendicular to a line with equation $ax + by + c = 0$, $(ab \neq 0)$. We observe that the slope m of L must be such that

$$-\frac{a}{b}m = -1 , \text{ or } m = \frac{b}{a} .$$

Thus L must have the equation in point-slope form,

$$y - y_0 = \frac{b}{a}(x - x_0) .$$

This is equivalent to

$$(3) \qquad bx - ay - (bx_0 - ay_0) = 0 .$$

If we substitute the specific values for a , b , x_0 , and y_0 in this general equation, we obtain

$$3x - 2y - (3 \cdot 4 - 2 \cdot 3) = 0 \text{, or } 3x - 2y - 6 = 0 .$$

If we generalize the notion of angle so that we may speak meaningfully of the measure of the "angle" between two parallel lines, we may obtain both these results as corollaries to the more general problem of determining the angle between two lines. Let two parallel directed lines have the same sense of direction. Then the projection of each positive ray of one line on the second line is also a ray and coincides with a positive ray of the second line. The coincident rays form angles whose measure is 0° or 0 radians. When two parallel directed lines have opposite senses of direction, the projection of each positive ray of one line on the second line is also a ray, but in this case, it is opposite to a positive ray of the second line. The pairs of opposite rays form angles whose measure is 180° or π radians. We speak of _parallel_ and _antiparallel directed lines_ respectively to distinguish between these two cases.

The preceding discussion suggests the following conventions. The _measure of the angle between two parallel directed lines_ is said to be 0° or 0 radians. The _measure of the angle between two antiparallel lines_ is said to be 180° or π radians.

Although the Law of Cosines was not developed for angles of measure 0° or 180° , the relationship it describes is still valid. We shall leave the justification as an exercise. If this extension is made, we may apply Equation (1) to parallel and antiparallel directed lines. In these cases, equivalent conditions are that $\cos\theta = 1$ and $\cos\theta = -1$ respectively. Thus, if the lines are parallel, $\cos\theta = \pm 1$ and Equation (1) becomes

$$\frac{1 + m_1 m_2}{\sqrt{1 + m_1^2}\sqrt{1 + m_2^2}} = \pm 1 .$$

This is equivalent to

$$(1 + m_1 m_2)^2 = (1 + m_1^2)(1 + m_2^2) ,$$

or

$$1 + 2m_1 m_2 + m_1^2 m_2^2 = 1 + m_1^2 + m_2^2 + m_1^2 m_2^2 .$$

This becomes

$$m_1^2 - 2m_1m_2 + m_2^2 = 0 ,$$

or

$$(m_1 - m_2)^2 = 0 ,$$

which is true if and only if $m_1 = m_2$. Thus, nonvertical lines are parallel if and only if

(4) $\cos \theta = \pm 1$, which is equivalent to $m_1 = m_2$.

Thus, we may express the condition that two nonvertical lines are parallel either in terms of the angle between them or in terms of their slopes.

Example 3. Write an equation in general form for

(a) the line containing the point $(1,2)$ and parallel to the line $L = \{(x,y) : 3x - 2y + 6 = 0\}$, and

(b) the line containing (x_0,y_0) and parallel to the line $L = \{(x,y) : ax + by + c = 0 , \text{ where } b \neq 0\}$.

Solutions.

(a) The slope of both lines must be $\frac{3}{2}$, so the required line must have as an equation in point-slope form,

$$y - 2 = \frac{3}{2}(x - 1) .$$

This is equivalent to

$$2y - 4 = 3x - 3 , \text{ or } 3x - 2y + 1 = 0 .$$

(b) The slope of both lines must be $-\frac{a}{b}$, so the required line must have as an equation in point-slope form,

$$y - y_0 = -\frac{a}{b}(x - x_0) .$$

This is equivalent to

$$by - by_0 = -ax + ax_0 ,$$

or

(5) $$ax + by - (ax_0 + by_0) = 0 .$$

Since equations representing lines are frequently given in general form, we write an equivalent expression to Equation (1) for the cosine of the angle between two lines in terms of the coefficients in the equations.

Let two nonvertical lines L_1 and L_2 have respective slopes m_1 and m_2 and be represented by the equations

$$a_1x + b_1y + c_1 = 0 \text{ , where } a_1^2 + b_1^2 \neq 0 \text{ ,}$$

and

$$a_2x + b_2y + c_2 = 0 \text{ , where } a_2^2 + b_2^2 \neq 0 \text{ .}$$

We have observed that

$$m_1 = -\frac{a_1}{b_1} \text{ and } m_2 = -\frac{a_2}{b_2} \text{ .}$$

If we substitute these values in Equation (1), we obtain

$$\cos\theta = \frac{1 + \frac{a_1a_2}{b_1b_2}}{\sqrt{1 + \frac{a_1^2}{b_1^2}}\sqrt{1 + \frac{a_2^2}{b_2^2}}}$$

which is equivalent to

$$\cos\theta = \frac{\frac{a_1a_2 + b_1b_2}{b_1b_2}}{\sqrt{\frac{a_1^2 + b_1^2}{b_1^2}}\sqrt{\frac{a_2^2 + b_2^2}{b_2^2}}} = \frac{\frac{a_1a_2 + b_1b_2}{b_1b_2}}{\frac{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}}{b_1b_2}} \text{ ,}$$

or

(6) $$\cos\theta = \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}} \text{ .}$$

Since $a_1^2 + b_1^2 \neq 0$ and $a_2^2 + b_2^2 \neq 0$, Equation (6) is always defined. Furthermore, Equation (6) is valid even when L_1 or L_2 is vertical. We shall leave the justification as an exercise.

When two lines intersect, two pairs of vertical angles are formed. If the lines are not perpendicular, two of the angles are acute, while the other two are obtuse and supplementary to the acute angles. The cosine of an acute angle θ is positive, while its obtuse supplement $\angle\theta'$ is such that $\cos\theta' = -\cos\theta$. Thus, if we wish to obtain only the acute or right angle between lines L_1 and L_2, we consider

$$\cos\theta = \frac{|a_1a_2 + b_1b_2|}{\sqrt{a_1^2 + b_1^2}\,\sqrt{a_2^2 + b_2^2}}. \tag{7}$$

Example 4. Find the measure of the acute angle between $L_1 = \{(x,y) : 2x - 7y + 25 = 0\}$ and $L_2 = \{(x,y) : 3x - 2y - 5 = 0\}$.

Solution.

$$\cos\theta = \frac{|2\cdot 3 + (-7)(-2)|}{\sqrt{2^2 + (-7)^2}\cdot\sqrt{3^2 + (-2)^2}} = \frac{20\sqrt{53\cdot 13}}{53\cdot 13} \approx .762,$$

and $\theta \approx 40^\circ$.

Example 5. Let (ℓ_1, m_1) and (ℓ_2, m_2) be pairs of direction numbers for lines L_1 and L_2 respectively. Show that L_1 is perpendicular to L_2 if and only if $\ell_1\ell_2 + m_1m_2 = 0$.

Solution. This suggests a special case of Equation (6),

$$\cos\theta = \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\,\sqrt{a_2^2 + b_2^2}},$$

where a_1,b_1 and a_2,b_2 are the coefficients in general forms of equations for L_1 and L_2 respectively. We are considering perpendicularity, which is equivalent to $\cos \theta = 0$ or the condition

$$a_1a_2 + b_1b_2 = 0 .$$

We have already observed that $(-b,a)$ are direction numbers for a line $L = \{(x,y) : ax + by + c = 0 , \text{ where } a^2 + b^2 \neq 0\}$. This is true in general, as we shall ask you to justify in the exercises. Thus, we may write $a_1 = k_1m_1$, $b_1 = - k_1\ell_1$, $a_2 = k_2m_2$, and $b_2 = -k_2\ell_2$, where k_1 and k_2 are constants such that $k_1{}^2 + k_2{}^2 \neq 0$. We substitute these in the necessary and sufficient condition above to obtain

$$k_1m_1 \cdot k_2m_2 + (-k_1\ell_1)(-k_2\ell_2) = 0 ,$$

which is equivalent to

$$\ell_1\ell_2 + m_1m_2 = 0 . \tag{8}$$

Since the three equations are equivalent, both the statement and its converse follow.

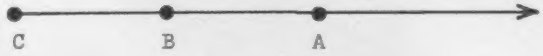
Exercises 2-7

1. Show that the relationship described by the Law of Cosines

$$\big(d(A,B)\big)^2 = \big(d(A,C)\big)^2 + \big(d(B,C)\big)^2 - 2d(A,C)d(B,C) \cos C$$

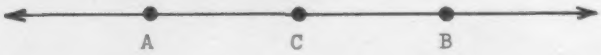
is also valid in the cases illustrated by

(a)



and

(b)



That is, justify the use of the Law of Cosines with angles of measure 0° and 180° .

2. Show that Equation (6) in the text is valid when

   (a) one line is vertical. (Let $L_1 = \{(x,y) : a_1x + c_1 = 0 , a_1 \neq 0\}$ and $L_2 = \{(x,y) : a_2x + b_2y + c_2 = 0 , a_2^2 + b_2^2 \neq 0\}$)

   (b) both lines are vertical. (Let $L_1 = \{(x,y) : a_1x + c_1 = 0 , a_1 \neq 0\}$ and $L_2 = \{(x,y) : a_2x + c_2 = 0 , a_2 \neq 0\}$.)

3. Which, if any, of the lines with the given equations are parallel? perpendicular? the same line?

   $L_1 : 3x - 4y = 12$

   $L_2 : y = \frac{4}{3}x - 3$

   $L_3 : 8x + 6y - 15 = 0$

   $L_4 : \frac{x}{4} - \frac{y}{3} = 1$

   $L_5 : \frac{x - 3}{-6 - 3} = \frac{y - 1}{-11 - 1}$

4. Find an angle between each of the pairs of lines with the given equations.
   - (a) $2x - 3y + 1 = 0 , x - 2y + 3 = 0$
   - (b) $x + 2y + 3 = 0 , y = 2x - 4$
   - (c) $y = 3 , x + y = 7$
   - (d) $3x + 2y + 5 = 0 , x - 2y + 5 = 0$
   - (e) $y = 2x - 5 , 4x - 2y + 7 = 0$
   - (f) $y = 2 , x = 3$

5. If $P = (a,b)$, $Q = (-b,a)$, and $a^2 + b^2 \neq 0$, show that $\overleftrightarrow{OP} \perp \overleftrightarrow{OQ}$.

6. Let $L_1 = \{(x,y) : 2x - 3y + 4 = 0\}$ and $L_2 = \{(x,y) : 3x + y - 2 = 0\}$. Write an equation in general form of a line L_3 which is:
   - (a) $\parallel L_1$ and contains the origin.
   - (b) $\parallel L_2$ and contains the point $(1,5)$.
   - (c) $\perp L_1$ and contains the point $(3,4)$.
   - (d) $\perp L_2$ and contains the point $(2,-1)$.

7. Find an equation for a line meeting the following conditions:
   - (a) Parallel to $L = \{(x,y) : 2x - 5y + 7 = 0\}$ and containing $P_1 = (2,7)$
   - (b) Perpendicular to $L = \{(x,y) : 3x + 2y - 1 = 0\}$, containing $(2,7)$.
   - (c) The perpendicular bisector of $\overline{AB}$, if $A = (-3,2)$ and $B = (5,-1)$.
   - (d) Parallel to the x-axis and containing $P_1 = (5,7)$.
   - (e) Parallel to the y-axis and containing $P_1 = (5,7)$.

8. Quadrilateral ABCD is a parallelogram. Find the coordinates of D if $A = (1,2)$, $B = (5,7)$, $C = (8,-3)$. If the order of the vertices of the parallelogram were not specified, how many possibilities would there be for D ?

9. A line L_1 makes an angle whose cosine is $\frac{3}{10}\sqrt{10}$ with $L_2 = \{(x,y) : 3x - y + 5 = 0\}$. What is the slope of L_1 ? Find its equation if it contains the point $(1,-2)$.

10. Let $A = (5,1)$, $B = (-2,3)$, and $C = (-3,4)$.

(a) Write the equations of $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{CA}$ in general form.

(b) What is the slope of each of these lines?

(c) Find the measures of the three angles of triangle ABC .

(d) Write equations of the lines containing the altitudes of triangle ABC in general form.

11. Let $L_1 = \{(x,y) : a_1x + b_1y + c_1 = 0$, where $a_1^2 + b_1^2 \neq 0\}$ and
$L_2 = \{(x,y) : a_2x + b_2y + c_2 = 0$, where $a_2^2 + b_2^2 \neq 0\}$.

Let L_1' be perpendicular to L_1 and contain the origin and let L_2' be perpendicular to L_2 and contain the origin.

(a) Write equations for L_1' and L_2' in general form.

(b) If L_1 and L_2 form an $\angle\theta$, prove that there is an $\angle\phi$, formed by L_1' and L_2' , such that $\cos\phi = \cos\theta$.

(c) Interpret the results of Part (b) in words.

12. Show that if lines L_1 and L_2 have pairs of direction cosines (λ_1,μ_1) and (λ_2,μ_2) respectively, then

(a) $\lambda_1\lambda_2 + \mu_1\mu_2 = \cos\theta$, where $\angle\theta$ is an angle formed by L_1 and L_2 ,

(b) $|\lambda_1\lambda_2 + \mu_1\mu_2| = \cos\theta$, where $\angle\theta$ is the least angle formed by L_1 and L_2 , and

(c) $\lambda_1\lambda_2 + \mu_1\mu_2 = 0$ if and only if L_1 and L_2 are perpendicular.

2-8. Normal and Polar Forms of an Equation of a Line.

In this section we shall introduce forms of an equation of a line which display the geometric properties discussed in the last section. We shall also consider a related expression for the distance between a point and a line.

Normal Form. The results of Example 5 in Section 2-6 suggest another characterization of a line in a plane. This characterization leads to yet another form of an equation of a line; the form has several useful applications.

Once a rectangular coordinate system has been defined in a plane, any directed segment $\overrightarrow{OP}$, emanating from the origin and terminating at another point P in the plane, is determined by the distance $d(O,P)$ and the direction cosines, $\cos\alpha = \lambda$ and $\cos\beta = \mu$, of the ray $\overrightarrow{OP}$. In the plane any line L which does not contain the origin may be described simply as the set of points which is perpendicular, or normal, to the directed segment $\overrightarrow{OP}$ at P. The directed segment $\overrightarrow{OP}$ is also said to be normal to L, and is called the normal segment of L. The distance $d(O,P)$ is called the normal distance of L (and is, of course, the distance from O to L).

In Figure 2-16 we let $\overrightarrow{OP_0}$ be the normal segment of L and let $p = d(O,P_0)$. Then $P_0 = (p\cos\alpha, p\cos\beta) = (p\lambda, p\mu)$. Now $(p\lambda, p\mu)$ is also a pair of direction numbers for the line $\overleftrightarrow{OP_0}$. If $p = (x,y)$ is any point of L other than P_0, $(x - p\lambda, y - p\mu)$ is a pair of direction numbers for L.

As we have seen in Example 5 of Section 2-7, L is normal to $\overleftrightarrow{OP_0}$ at P_0 if and only if $p\lambda(x - p\lambda) + p\mu(y - p\mu) = 0$. We note that the coordinates of the point P_0 also satisfy this equation.

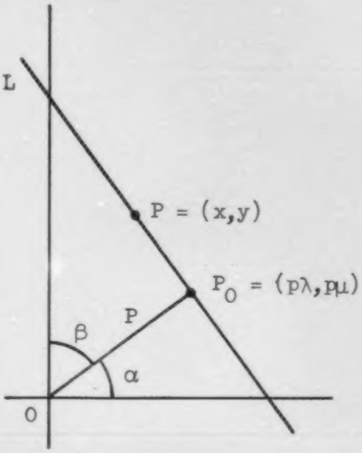


Figure 2-16

The equation is equivalent to

$$\lambda x + \mu y - p(\lambda^2 + \mu^2) = 0 .$$

Since $\lambda^2 + \mu^2 = 1$, this may be written as

$$\lambda x + \mu y - p = 0 , \tag{1}$$

which is called a normal form of an equation of a line. We cannot stress too strongly that in this form λ and μ are not direction cosines of the line itself, but of the normal segment. The constant p is always positive and is the distance between the origin and the line.

We may always express an equation of a line in general form; Example 5 in Section 2-6 also suggests how we may find the normal form of an equation of a line L which does not contain the origin. Let $L = \{(x,y) : ax + by + c = 0,$ where $(a^2 + b^2)c \neq 0\}$. The normal form of such an equation is a special case of the general form. Both are linear equations, and two linear equations are equivalent if and only if their corresponding coefficients are proportional. Thus, the pair (a,b) is equivalent to the normalized pair (λ,μ) of direction numbers for the normal segment. Consequently, (a,b) is a pair of direction numbers for the normal segment and

$$(\lambda,\mu) = \left(\frac{a}{\sqrt{a^2 + b^2}} , \frac{b}{\sqrt{a^2 + b^2}}\right) \text{ or } \left(\frac{-a}{\sqrt{a^2 + b^2}} , \frac{-b}{\sqrt{a^2 + b^2}}\right).$$

Our choice between these two possibilities is determined by the requirement that $p > 0$. If $c < 0$ in the equation $ax + by + c = 0$, we divide by $\sqrt{a^2 + b^2}$ to obtain the normal form; if $c > 0$, we divide by $-\sqrt{a^2 + b^2}$.

Example 1. Write $3x - 4y + 12 = 0$ in normal form.

Solution. Since the constant term is positive, we divide by $-\sqrt{3^2 + (-4)^2} = -5$ to obtain

$$-\frac{3}{5}x + \frac{4}{5}y - \frac{12}{5} = 0 .$$

We see from the equation that the normal distance is $\frac{12}{5}$, $\cos \alpha = -\frac{3}{5}$, and $\cos \beta = \frac{4}{5}$.

Example 2. Put the equation $-6x - 5y - 20 = 0$ in normal form.

Solution.
$$-\frac{6}{\sqrt{61}}x - \frac{5}{\sqrt{61}}y - \frac{20}{\sqrt{61}} = 0 .$$

We have not considered lines containing the origin. In the general form of an equation for such a line L, c is zero. There is no directed segment normal to the line emanating from the origin, nor is there a unique standard procedure in this case. Some mathematicians hold that there are two normal forms corresponding to the normal rays $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ as illustrated in Figure 2-17; others prefer a unique form corresponding to the normal ray for which $0 \leq \alpha < 180^\circ$ and $0 \leq \beta \leq 90^\circ$. In the first case we obtain a normal form by dividing a general form with $c = 0$ by either $\sqrt{a^2 + b^2}$ or $-\sqrt{a^2 + b^2}$; in the second case, we obtain a unique normal form by dividing by $\sqrt{a^2 + b^2}$ when $b > 0$, by $-\sqrt{a^2 + b^2}$ when $b < 0$, and by a when $b = 0$. You may follow either convention.

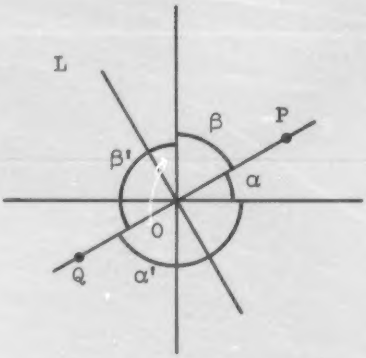


Figure 2-17

Example 3. Find the normal forms of equations of the lines

(a) $L_1 = \{(x,y) : 3x + 4y = 0\}$.

(b) $L_2 = \{(x,y) : 3x - 2y = 0\}$.

(c) $L_3 = \{(x,y) : -2x = 0\}$.

Solution.

(a) Alternate forms: $\frac{3}{5}x + \frac{4}{5}y = 0$ or $-\frac{3}{5}x - \frac{4}{5}y = 0$

Unique form: $\frac{3}{5}x + \frac{4}{5}y = 0$.

(b) Alternate forms: $\frac{3}{\sqrt{13}}x - \frac{2}{\sqrt{13}}y = 0$ or $-\frac{3}{\sqrt{13}}x + \frac{2}{\sqrt{13}}y = 0$

Unique form: $-\frac{3}{\sqrt{13}}x + \frac{2}{\sqrt{13}}y = 0$

(c) Alternate forms: $x = 0$ or $-x = 0$

Unique form: $x = 0$.

A useful application related to the normal form is to find the distance between a point $P_1 = (x_1,y_1)$ and a line $L = \{(x,y) : \lambda x + \mu y - p = 0\}$.

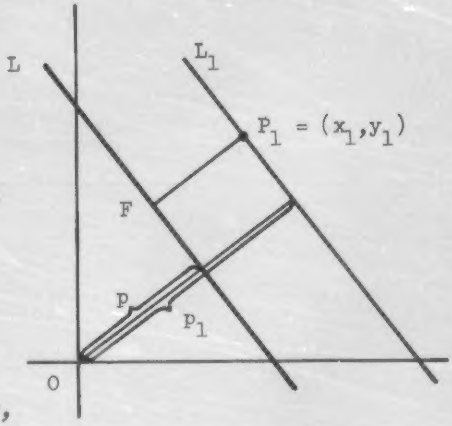


Figure 2-18

We illustrate this situation in Figure 2-18. F is the projection of P_1 onto L and we wish to find $d(P_1,F)$. There exists a unique line L_1 which is parallel to L and which contains P_1. L_1 is represented by the equation $\lambda x + \mu y - p_1 = 0$. Since L_1 contains (x_1,y_1), $\lambda x_1 + \mu y_1 - p_1 = 0$ or $p_1 = \lambda x_1 + \mu y_1$.

There are several cases to consider, including the following two:

i) O and P_1 are on opposite sides of L as in Figure 2-18. In this case, $d(P_1,F) = p_1 - p = \lambda x_1 + \mu y_1 - p$.

ii) P_1 is on the same side of L as O; P_1 is farther than O from L. In this case, the normal segment to L_1 has the opposite sense of direction and its direction cosines are $-\lambda, -\mu$. Hence, its normal distance is $-\lambda x_1 - \mu y_1$, or $-p_1$, and

$$d(P_1,F) = p + (-p_1) = |\lambda x_1 + \mu y_1 - p| .$$

You may find it helpful to draw a figure to illustrate the second situation.

We leave the other possibilities as an exercise. In each case the distance d between the point $P_1 = (x_1,y_1)$ and the line $L = \{(x,y) : \lambda x + \mu y - p = 0\}$ is given by

$$d = |\lambda x_1 + \mu y_1 - p| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} . \tag{2}$$

Example 3. Find the distance between $P = (3,-10)$ and $L = \{(x,y) : 3x - 4y + 12 = 0\}$.

Solution. From Equation (2) we obtain

$$d = \frac{|3(3) - 4(-10) + 12|}{\sqrt{3^2 + (-4)^2}} = \frac{61}{5} = 12.2 .$$

Polar Form. The analytic representation of a line in a plane with a polar coordinate system is similar to the normal form.

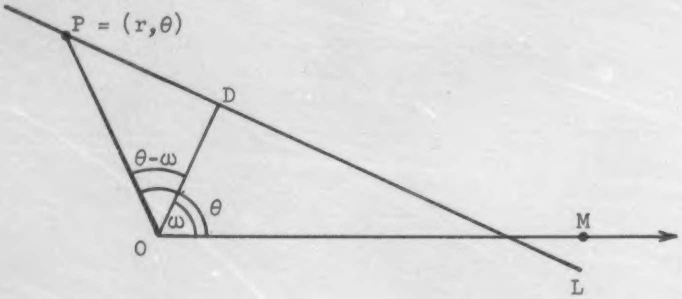


Figure 2-19

In Figure 2-19 we illustrate a line L in a plane with a polar coordinate system. Let $\overrightarrow{OD}$ be the normal segment to L, let p be the normal distance, and let $\angle\omega$ be the polar angle of D. If $P = (r,\theta)$ is any point of L other than D, then in right triangle ODP we have

$$r\cos(\theta - \omega) = p\,, \tag{3}$$

which is called the polar form of an equation of a line which does not contain the pole. We note that $D = (p,\omega)$ satisfies Equation (3) and that, since $\cos(\omega - \theta) = \cos(\theta - \omega)$, the equation is valid for points whose polar angle has measure θ which is less than ω.

Points are on a line L containing the pole if and only if they may all be described by the same or equivalent polar angles. Thus, the representations of a line containing the pole are

$$L = \{(r,\theta) : \theta = k + n\pi\,, \text{ where } k \text{ is real and } n \text{ is an integer}\}\,,$$

or

$$L = \{(r,\theta) : \theta = k + 180n^\circ\,, \text{ where } k \text{ is real and } n \text{ is an integer}\}\,.$$

The appearance of the degree symbol in the second representation does not mean that the right-hand member of the equation does not represent a simple real number; rather, it is a convention to indicate that the angle is measured in degrees.

Example 4.

(a) Find a polar form of an equation of the line with inclination 135° and whose distance from the pole is 2.

(b) Find a polar equation for a line containing the pole with inclination 60°.

Solution.

(a) If the line intersects the polar axis, the polar angle of the normal segment is $\frac{\pi}{4}$, and the polar form of an equation is

$$r\cos(\theta - \tfrac{\pi}{4}) = 2 .$$

If the line intersects the ray opposite to the polar axis, the polar angle of the normal segment is $\frac{5\pi}{4}$, and the polar form of an equation is

$$r\cos(\theta - \tfrac{5\pi}{4}) = 2 .$$

(b) The line has polar equations

$$\theta = \frac{\pi}{3} + n\pi \text{, where } n \text{ is an integer,}$$

or

$$\theta = 60^\circ + 180n^\circ \text{, where } n \text{ is an integer.}$$

If a line has already been represented in a rectangular coordinate system as

$$L = \{(x,y) : ax + by + c = 0 ,\ a^2 + b^2 \neq 0\} ,$$

we may obtain a polar equation in the related polar coordinate system simply by substitution from the relations $x = r\cos\theta$ and $y = r\sin\theta$. The equation becomes

(4) $$a\,r\cos\theta + b\,r\sin\theta + c = 0 \text{, where } a^2 + b^2 \neq 0 .$$

In order to see how this equation is related to the usual polar form, we recall that $ax + by + c = 0$ has the equivalent normal form $\lambda x + \mu y - p = 0$, with the corresponding coefficients proportional. Furthermore, $\lambda = \cos\alpha$ and $\mu = \cos\beta$, where $\angle\alpha$ and $\angle\beta$ are the direction angles of the normal segment. In the polar coordinate system which we have assumed to relate the coordinates, we let $\angle\omega$ be a polar angle which contains the normal segment of L. Thus $\omega = \pm\alpha$ and $\cos\omega = \cos\alpha = \lambda$. Furthermore,

$\sin\omega = \cos\beta = \mu$. If you have worked Exercise 7 of Section 2-6, you should already be aware that this is true; otherwise, you should justify now that it is so.

Let $\lambda x + \mu y - p = 0$ be the normal form of Equation (4). We substitute for λ , μ , x , and y to obtain

$$\cos\omega \cdot r\cos\theta + \sin\omega \cdot r\sin\theta - p = 0 ,$$

or

$$r(\cos\theta\cos\omega + \sin\theta\sin\omega) = p ,$$

which is equivalent to

$$r\cos(\theta - \omega) = p .$$

Example 5. Assume the usual orientation of the polar axis and find the polar form of an equation of the line

(a) 2 units to the right of the pole and perpendicular to the polar axis,

(b) 3 units above the pole and parallel to the polar axis,

(c) 1 unit to the left of the pole and perpendicular to the polar axis,

(d) 4 units below the pole and parallel to the polar axis.

(e) $L = \{(x,y) : x + \sqrt{3}y - 12 = 0\}$.

Solution.

(a) Since the length and polar angle of the normal segment are 2 and 0 respectively, the polar form of an equation is $r\cos\theta = 2$.

(b) $r\cos(\theta - \frac{\pi}{2}) = 3$. A simpler equation is $r\sin\theta = 3$.

(c) $r\cos(\theta - \pi) = 1$. Another equation is $r\cos\theta = -1$.

(d) $r\cos(\theta - 270^\circ) = 4$. Another equation is $r\sin\theta = -4$.

(e) $x + \sqrt{3}y - 12 = 0$ is equivalent to the normal form

$$\frac{1}{2}x + \frac{\sqrt{3}}{2}y - 6 = 0 ,$$

and the corresponding polar equation

$$\frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta - 6 = 0 ,$$

or

$$r\left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right) = 6 . \tag{5}$$

If we let $\frac{1}{2} = \cos\omega$ and $\frac{\sqrt{3}}{2} = \sin\omega$, we obtain $\frac{\pi}{3}$ as a suitable value for ω. We substitute in Equation (5) to obtain

$$r\left(\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta\right) = 6 ,$$

$$r\cos\left(\frac{\pi}{3} - \theta\right) = 6 ,$$

or

$$r\cos\left(\theta - \frac{\pi}{3}\right) = 6 ,$$

which is in polar form.

Example 6. Assume the usual relationship between the polar axis and the x- and y-axes and write an equivalent equation in rectangular coordinates for

$$r\cos(\theta - \omega) = p .$$

Solution. If we expand $\cos(\theta - \omega)$, we obtain the equation

$$r\cos\theta\cos\omega + r\sin\theta\sin\omega = p .$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, this is equivalent to

$$x\cos\omega + y\sin\omega = p . \tag{6}$$

Because $\cos\omega = \lambda$ and $\sin\omega = \mu$, Equation (6) is sometimes called the normal form of an equation of a line.

Exercises 2-8

1. Write each of the following equations in normal form:

(a) $4x - 3y + 15 = 0$

(b) $5x + 12y - 65 = 0$

(c) $3x - 2y - 6 = 0$

(d) $5y - 3x + 12 = 0$

(e) $y = 3x - 7$

(f) $y = -\frac{8}{15}x + 2$

(g) $12x - 5y = 0$

(h) $7y = 20$

(i) $9x + 15 = 0$

(j) $\frac{x}{12} - \frac{y}{5} = 1$

(k) $\frac{y}{8} - \frac{x}{15} = 1$

(l) $y - 2 = \frac{3}{4}(x - 5)$

2. For Parts (a) and (b) of Exercise 1, draw the normal segment by using the information concerning α, β, and p which is supplied by the equation. Then draw the line perpendicular to the normal segment at its terminal point. Verify that this is the line represented by the given equation.

3. Without using rectangular coordinates write in polar form the equation of a line

   (a) which is parallel to the polar axis and 4 units above it.

   (b) which is perpendicular to the polar axis and 4 units to the right of the pole.

   (c) through the pole with slope $\sqrt{3}$.

   (d) which contains the point $(-3,135^\circ)$ and has inclination 45°.

   (e) which contains the point $(3,0)$ and has inclination 30°.

   (f) which contains the point $(2,\frac{\pi}{4})$ and has inclination 45°.

   (g) which is perpendicular to the line with equation $r\cos(\theta - \frac{\pi}{3}) = 2$ and contains the point $(4,\frac{\pi}{2})$.

   (h) which is parallel to the line with equation $r\cos(\theta - \frac{\pi}{4}) = 1$ and contains the point $(2,-135^\circ)$.

4. Transform each of the following equations to polar form.

   (a) $x - 4 = 0$
   (b) $y + 4 = 0$
   (c) $x = 0$
   (d) $x + y + 2 = 0$
   (e) $3x - 2y + 6 = 0$
   (f) $x + \sqrt{3}y - 2 = 0$
   (g) $15y - 8x + 34 = 0$

5. Let $L = \{(x,y) : \lambda x + \mu y - p = 0$, where $\lambda^2 + \mu^2 = 1\}$ and let $P_1 = (x_1,y_1)$. Show that the distance between P_1 and L is $|\lambda x_1 + \mu y_1 - p|$ when

   (a) P_1 is on L.

   (b) P_1 is on the same side of L as the origin O; P_1 is closer than O to L.

   (c) P_1 is on the same side of L as O; P_1 and O are equidistant from L.

6. Find the distance between P and L :

   (a) $P = (6,8)$; $L = \{(x,y) : 12x - 5y + 26 = 0\}$.

   (b) $P = (-3,2)$; $L = \{(x,y) : 3x - 4y - 5 = 0\}$.

   (c) $P = (-5,-7)$; $L = \{(x,y) : y = 4x - 7\}$.

   (d) $P = (4,-5)$; $L = \{(x,y) : \frac{x}{7} + \frac{y}{5} = 1\}$.

   (e) $P = (8,11)$; $L = \{(x,y) : y - 4 = \frac{7}{5}(x - 3)\}$.

7. Find equations of the lines bisecting the angles formed by the lines $L_1 = \{(x,y) : 3x - 4y + 5 = 0\}$ and $L_2 = \{(x,y) : 12x + 5y - 13 = 0\}$.

   Hint: How is an angle bisector described as a locus?

8. Find equations of the lines bisecting the angles formed by $L_1 = \{(x,y) : 3x - 4y + 12 = 0\}$ and $L_2 = \{(x,y) : 12x - 5y - 60 = 0\}$.

   (See Exercise 7.)

9. Find equations of the lines bisecting the angles formed by the lines $L_1 = \{(x,y) : \lambda_1 x + \mu_1 y - p_1 = 0 \, , \, \lambda_1^2 + \mu_1^2 = 1\}$ and $L_2 = \{(x,y) : \lambda_2 x + \mu_2 y - p_2 = 0 \, , \, \lambda_2^2 + \mu_2^2 = 1\}$.

   (See Exercise 7.)

10. Write the equation $r \cos \theta - 3 = 0$ in rectangular coordinates.

11. Write the equation $x - y = 0$ in polar coordinates.

12. Write the equation $x^2 + y^2 = 36$ in polar coordinates.

13. Write the equation $r = 4 \cos \theta$ in rectangular coordinates.

    Hint: Multiply both members of the equation by r . Check that the pole is in the graph of the original equation. Explain why you must make this check.

14. Write the equation $r = 2a \cos \theta$ in rectangular coordinates.

    (See Exercise 13.)

15. Transform to rectangular form.

    (a) $\theta = 60^\circ$

    (b) $r \sin \theta + 4 = 0$

    (c) $r = 5$

16. Sketch the locus of each equation in Exercise 15.

17. (a) Transform $x^2 + y^2 - 4x = 0$ into polar coordinates.

(b) Transform $r = 5 \cos \theta - 3 \sin \theta$ into rectangular coordinates.

(c) Transform $r \cos(\theta - \frac{3\pi}{2}) = 4$ into rectangular coordinates.

(d) Transform $(x^2 + y^2 + y)^2 = x^2 + y^2$ into polar coordinates.

2-9. Summary.

In this chapter you have encountered many topics which were already familiar from various sources. Our hope is that by gathering them together, we have offered you not only the chance to refresh your memory, but also new insight into the coherence and application of these ideas.

We first considered the basis for coordinates on a line and the characterization of subsets of a line in terms of coordinates. Next we reviewed with care the rectangular coordinate system in the plane and various analytic representations of a line in the plane.

Polar coordinates may well be a concept new to you. Relations of both mathematical interest and physical importance may often be represented most simply by equations in polar coordinates.

We have stressed our freedom of choice in introducing coordinate systems. The ease of our solution of problems depends in part upon our foresight in establishing a framework of reference.

In problem solving the danger always exists that we might let the algebra do our thinking for us. A geometric interpretation will both guide and control our application of algebraic techniques. Throughout this chapter we have emphasized the roles of algebra and geometry in the interpretation of such concepts as congruence, betweenness, direction on a line, the measure of angles, and the measure of distance between points and lines.

In the next chapter we shall study vectors. Vectors form in themselves a bridge between geometry and algebra, for they are geometric objects for which algebraic operations are defined.

Review Exercises - Section 2-6 through Section 2-8

1. Find a pair of direction numbers, a pair of direction cosines, and a pair of direction angles for

   (a) the line containing the points $(-3,7)$ and $(4,-3)$.

   (b) a line with slope $\frac{24}{25}$.

   (c) a ray emanating from $(2,3)$ and containing $(-4,8)$.

   (d) the line $L = \{(x,y) : 6x - 7y + 4 = 0\}$.

   (e) $L = \{(x,y) : \frac{x - 2}{5 - 2} = \frac{y + 4}{-7 + 4}\}$.

   (f) $L = \{(x,y) : y = \frac{7}{4}x + 9\}$.

   (g) $L = \{(x,y) : \frac{x}{5} + \frac{y}{-10} = 1\}$.

   (h) $L = \{(x,y) : y + 2 = \frac{-1 + 2}{3 - 5}(x - 5)\}$.

2. In each part below determine whether the three points are collinear.

   (a) $(11,13)$, $(-4,1)$, and $(1,5)$.
   (b) $(1,-2)$, $(-5,7)$, and $(6,-12)$.
   (c) $(23,17)$, $(-1,-1)$, and $(-17,-13)$.
   (d) $(0,-4)$, $(-3,8)$, and $(5,-11)$.

In Exercises 3-8 let $A = (-3,1)$, $B = (2,5)$, $C = (4,-1)$.

3. Find the distances: $d(A,B)$, $d(A,C)$, $d(B,C)$.

4. Write in general form the equations of the three lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$.

5. Use the results of Exercise 4 to find the lengths of the three altitudes of $\triangle ABC$.

6. Use the results of Exercises 3 and 5 to find the area of $\triangle ABC$.

7. In $\triangle ABC$, find equations of

   (a) the line containing the bisector of $\angle A$.
   (b) the line containing the bisector of $\angle B$.
   (c) the line containing the bisector of $\angle C$.

In Exercises 8-11 , let $L_1 = \{(x,y) : 2x - 3y + 6 = 0\}$,
$L_2 = \{(x,y) : 3x + 4y - 12 = 0\}$,
and $L_3 = \{(x,y) : x - 2y + 4 = 0\}$.

8. Find the distance from

(a) A to each of the lines L_1 , L_2 , L_3 .

(b) B to each of the lines L_1 , L_2 , L_3 .

(c) C to each of the lines L_1 , L_2 , L_3 .

9. Find equations for the two angle bisectors of the angles formed by

(a) L_1 , L_2 .

(b) L_1 , L_3 .

(c) L_2 , L_3 .

10. Find the distances between the parallel lines:

(a) L_1 as above, and $L_4 = \{(x,y) : 2x - 3y + 12 = 0\}$.

(b) L_2 as above, and $L_5 = \{(x,y) : 3x + 4y - 1 = 0\}$.

(c) L_3 as above, and $L_6 = \{(x,y) : x - 2y + 10 = 0\}$.

11. Find two points on L_1 which are 5 units away from L_2 .

12. Find the angles between $L_1 = \{(x,y) : \frac{x-2}{5-2} = \frac{y-4}{3-4}\}$

and $L_2 = \{(x,y) : \frac{x-3}{4-3} = \frac{y-2}{4-2}\}$.

13. Show that $L_1 = \{(x,y) : \frac{x-3}{-2-3} = \frac{y-2}{5-2}\}$ is perpendicular to

$L_2 = \{(x,y) : \frac{x-1}{4-1} = \frac{y-4}{9-4}\}$.

14. Find the angles between L_1 and L_2 , where L_1 contains the points (3,4) and (-1,-1) , and L_2 contains the points (-4,6) and (3,0) .

15. Find the measure of the angle whose sides have pairs of direction cosines, $\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ and $\left(\frac{7}{\sqrt{58}}, \frac{3}{\sqrt{58}}\right)$ respectively.

16. Show that triangle ABC is a right triangle, where A = (3,4) , B = (-2,7) , and C = (6,9) .

17. Find the normal form of the equations

(a) $3x - 7y + 29 = 0$.

(b) $y = \frac{20}{21}x + 58$.

(c) $\frac{x}{6} + \frac{y}{-8} = 1$.

(d) $3x - 7y = 0$.

(e) $7 = 5x$.

18. Find the polar form of the equation of the line

(a) which intersects the polar axis at $(2,0)$ and has inclination $\frac{5\pi}{6}$.

(b) which is perpendicular to the polar axis at a point 4 units from the pole on the ray opposite to the polar axis.

(c) contains the pole and the point $(7,147^\circ)$.

19. Transform to rectangular coordinates:

(a) $r \cos(\theta - \frac{7\pi}{6}) = 5$

(b) $3r \sin \theta - 4r \cos \theta = 12$

20. Transform to polar coordinates:

(a) $\frac{x}{7} + \frac{y}{8} = 1$

(b) $y = \frac{8}{15}x - 12$

Challenge Exercises

For each of Exercises 1-6 write an equation to represent all lines,

1. parallel to $3x - 4y + 10 = 0$,
2. perpendicular to $3x - 4y + 10 = 0$,
3. containing the origin,
4. containing the point $(2,3)$,
5. containing the point $(4,0)$ and parallel to line in Exercise 1 ,
6. having slope -3 .
7. Prove analytically that the lines containing the bisectors of the angles formed by any two intersecting lines are perpendicular.

8. Prove: If $P_1 = (x_1,y_1)$ is not on $L = \{(x,y) : ax + by + c = f(x,y) = 0\}$, then $f(x,y) = f(x_1,y_1)$ is an equation of a line parallel to L.

In Exercises 9-13 let $A = (0,0)$, $B = (1,0)$, and $C = (a,b)$, where $b \neq 0$.

9. Prove that the lines containing the altitudes of triangles ABC are concurrent at a point H. Find the coordinates of H.

10. Prove that the lines containing the medians of triangle ABC are concurrent at a point G. Find the coordinates of G.

11. Prove that the lines containing the bisectors of the angles of triangle ABC are concurrent at a point I. Find the coordinates of point I.

12. Prove that the perpendicular bisectors of the sides of triangle ABC are concurrent at a point E. Find the coordinates of point E.

13. Prove that the points H, G, and E in Exercises 9, 10, and 12 collinear. Find an equation of the line containing them.

Chapter 3

VECTORS AND THEIR APPLICATIONS

3-1. Why Study "Vectors"?

The use of vectors is becoming increasingly important. For example, many of the problems regarding space travel and ordinary air travel on the earth are solved by vector methods.

Vectors were created by the mathematical physicists William R. Hamilton and Herman Grassman in about the middle of the nineteenth century to solve the many problems involving forces and motion. Since that time vectors have been applied in many branches of science, engineering, and mathematics. The work of Hamilton and Grassman was based on the earlier development of analytic geometry by René Descartes and Pierre Fermat in the seventeenth century.

Vector methods and the non-vector methods of analytic geometry are both widely used in proving geometric theorems and they have become so interwoven that it is at times impossible to separate them. In fact, several books have been published recently under titles such as "Analytic Geometry: A Vector Approach", and courses in calculus make extensive use of both vector and non-vector methods interchangeably. This is one of the principal reasons for including this chapter in our book--to give you an additional tool to apply to find interesting relations among geometric objects and to prove some geometric theorems. An additional reason is the future need in scientific or engineering studies or in mathematics courses.

To understand what follows you should recall what you learned in your course in geometry. If you have studied about vectors before, part of this material will serve as a review and you may be interested in comparing the two approaches to the subject. However, no knowledge of vectors is assumed.

3-2. Directed Line Segments and Vectors.

In Chapter 2 we encountered directed line segments, which possess both direction and magnitude. A simple example of this geometric concept is that of a motion or displacement along a line. Let us say a boy starts at a given point and walks two miles. We don't know much about his trip until we are told the direction in which he walks or the point at which he ends. A displacement can then be represented in one of two ways:

(a) By a directed segment extending a given distance in a given direction from a given point.

(b) By a pair of points, one identified as the starting or *initial* point, the other as the ending or *terminal* point.

The symbol $\overrightarrow{AB}$ is used to denote such a directed line segment whose initial point is A and whose terminal point is B.

DEFINITION. By the *magnitude* of the *directed line segment* $\overrightarrow{AB}$ we mean $d(A,B)$, the length of the associated segment $\overline{AB}$.

We now turn our attention to the concept of a *vector*, which is closely related to the geometric concept of a directed line segment. Vectors were created by physicists to deal with concepts such as force, acceleration, velocity, flow of heat, and flow of electricity.

To understand this new concept, we need the following definition:

DEFINITION. *Directed line segments* will be considered *equivalent* *if* and only if they

(1) lie on the same or parallel lines,

(2) have the same sense of direction, and

(3) have the same magnitude.

For convenience, we shall use the term "parallel" in the sense of statement (1). The phrase "if and only if" means that the statement and its converse are both true.

DEFINITION. The infinite set of directed line segments equivalent to any given directed line segment is called a *vector*.

To understand more fully the concept of a vector let us recall an analogy from arithmetic. Here we have an infinite set of equivalent fractions which represent the same quantity; e.g. $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{5}{10}, \frac{11}{22}, \ldots\}$. Such a set is called a rational number.

It is common in many texts to use the word vector to mean, not the whole set of equivalent directed line segments, but any single member of that set. When convenient, and when there is no ambiguity we will follow this procedure. When we use the word vector in this way, and say that two vectors are equal, we mean they are members of the same set of equivalent directed line segments. In the case of the representation of rational numbers, when we say $\frac{2}{4} = \frac{3}{6}$ we mean that these two fractions represent the same rational number. We shall represent a vector by any of its members and we shall denote such directed line segments by $\overset{\leftharpoonup}{a}$, $\overset{\leftharpoonup}{b}$, $\overset{\leftharpoonup}{c}$, ...

Each rational number has a representative which is considered the "simplest", and that is the member whose numerator and denominator have no common factor. In the example above, $\frac{1}{2}$ is the simplest representative of the rational number.

In the same way, it will be convenient to have a "simplest" representative for each vector. For this purpose we require a reference point in space which we shall call the origin. Any point in space can serve as the origin, and to emphasize this freedom, we state the following principle:

> ORIGIN PRINCIPLE: Vectors may be related to any point in space as an origin.

The usefulness of this principle will become evident when vectors are applied to the solution of problems.

After an origin is selected in space, each vector (or equivalent set of directed line segments) contains a unique member with this origin as its initial point. We shall call this member the origin-vector and it will serve as the "simplest" representative of the vector. The symbol $\overset{\leftharpoonup}{A}$ will be the origin-vector representation for the vector $\overset{\leftharpoonup}{a}$, $\overset{\leftharpoonup}{B}$ for $\overset{\leftharpoonup}{b}$, ... as shown in Figure 3-1. Note that to each point A of the plane there now corresponds a unique origin-vector $\overset{\leftharpoonup}{A}$.

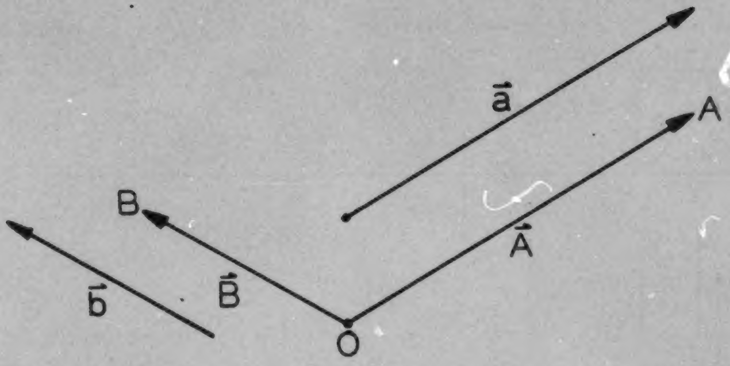


Figure 3-1

It is important to note that we do not always wish to use the simplest representative. For example, in adding $\frac{1}{2}$ and $\frac{1}{3}$, we find it most convenient to use the member $\frac{3}{6}$ instead of $\frac{1}{2}$ and $\frac{2}{6}$ instead of $\frac{1}{3}$. Likewise, in dealing with vectors, we shall frequently find it more convenient to use a representative of its set other than the origin-vector.

Vectors are very frequently associated with real numbers. In discussions involving vectors, real numbers will be referred to as scalars. The scalar which is the length of $\vec{a}$ will be denoted by $|\vec{a}|$ and will be referred to as its magnitude or absolute value. Other examples of scalars are the measures of angle, area, mass, and temperature. You will find it helpful to compare these with the examples of vectors given earlier.

DEFINITIONS. Any origin corresponds to an object called the zero vector and is denoted by $\vec{0}$.

A vector of unit length is called a unit vector. Note that

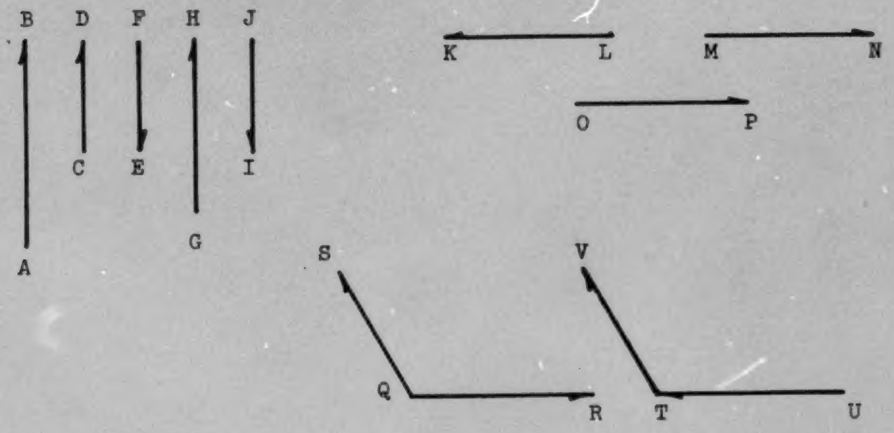
$\frac{a}{|a|}$ is the unit vector along $\vec{a}$.

Note also that the zero vector has zero magnitude but no particular direction. A unit vector exists in every direction.

Exercises 3-2

1. Draw a vector from (3,2) as defined in this chapter and indicate its simplest representative.

2. For the figures below indicate the sets of equivalent directed line segments.

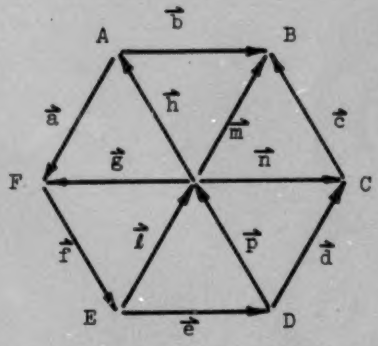

B
D
F
H
J
C
E
I
G
A
K
L
M
N
O
P
S
Q
R
V
T
U

3. Given the vertices A , B , C , and D of a parallelogram. List all the directed line segments determined by ordered pairs of these points. Which belong to the same vector?

4. Figure ABCDEF is a regular hexagon. In the diagram, find three replacements for $\vec{x}$ and $\vec{y}$ to make each of these statements true:

   (a) $\vec{x} = \vec{y}$

   (b) $\vec{x} = -\vec{y}$


A
$\vec{b}$
B
$\vec{a}$
$\vec{h}$
$\vec{m}$
$\vec{c}$
$\vec{g}$
$\vec{n}$
F
C
$\vec{f}$
$\vec{l}$
$\vec{p}$
$\vec{d}$
E
$\vec{e}$
D

5. Show the simplest representatives of four different unit vectors on a plane with a rectangular coordinate system. Do the same on a plane with a polar coordinate system.

6. List five geometric or physical concepts not listed in this section, which can be represented by vectors.

3-3. Sum and Difference of Vectors. Scalar Multiplication.

To get anything of either mathematical interest or physical usefulness, it is necessary to introduce operations on vectors. Since forces are conveniently represented by vectors, we may consider the problem of replacing two forces acting at a point by a single force called the resultant. A Dutch scientist, Simon Stevin (1548-1620) experimented with this problem and discovered that the resultant force could be represented by the diagonal of a parallelogram whose sides represented the original forces.

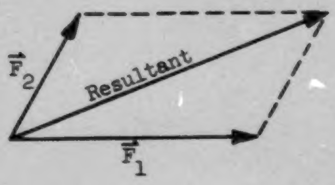


Figure 3-2

Thus a definition of vector addition is made which is consistent with observations of the physical world.

Before presenting such a definition, there is an important distinction to make between the use of origin-vectors and other vectors. You must be aware of this distinction.

We have already agreed in the statement of the "Origin Principle" that vectors may be related to any point in space as an origin. One reason for stating this principle is that it is more convenient to deal with origin-vectors when we seek a geometric interpretation.

We are about to define operations with vectors and prove several theorems. In order that the use of origin-vectors will not limit the application of the results we state the following principle:

ORIGIN-VECTOR PRINCIPLE. The sum and difference of vectors and the product of a vector by a scalar is equivalent to the sum, difference, and scalar product of their respective origin vectors.

There is one more significant statement to make in this regard. All proofs using origin-vectors depend in part upon the fact that all such vectors have a common initial point. The extension of such proofs to vectors in general can readily be made by choosing for any vectors those representatives which have a common initial point.

In other words, the algebraic relationships between vectors will hold in general, but the geometric interpretation must be limited to the geometric conditions assumed in the development.

DEFINITION.

(1) Let $\vec{P}$ and $\vec{Q}$ be two non-zero vectors not lying in the same line and with a common initial point 0. We define the vector sum of $\vec{P}$ and $\vec{Q}$, designated by $\vec{P} + \vec{Q}$, to be the unique vector with initial point 0 and whose terminal point is the vertex opposite 0 in the parallelogram formed with $\vec{P}$ and $\vec{Q}$ as sides.

(2) If $\vec{P}$ and $\vec{Q}$ have the same direction, $\vec{P} + \vec{Q}$ is the vector with the same direction, and with magnitude equal to the sum of the magnitudes of $\vec{P}$ and $\vec{Q}$. If $\vec{P}$ and $\vec{Q}$ have opposite directions, $\vec{P} + \vec{Q}$ is the vector with the same direction as the vector of larger magnitude, and with magnitude equal to the absolute value of the difference of the two magnitudes.

(3) For any vector $\vec{P}$, $\vec{P} + \vec{0} = \vec{0} + \vec{P} = \vec{P}$, where $\vec{0}$ denotes the zero vector.

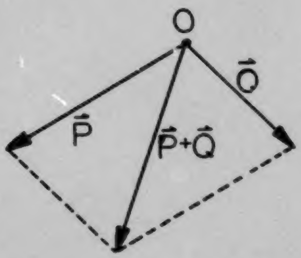


Figure 3-3

In arithmetic one usually considers multiplication as repeated addition of the same number. For example, $3 \times 2 = 2 + 2 + 2$. An analogous definition is made for the multiplication of a vector by a scalar. Thus $3\vec{A} = \vec{A} + \vec{A} + \vec{A}$. The second part of the above definition also tells us that $\vec{A} + \vec{A} + \vec{A}$ is a vector parallel to A , with the same sense of direction, and a magnitude three times as large. Generalizing this idea, one can state the following definition:

DEFINITION. Let r be a real number and $\vec{P}$ any vector. Then $r\vec{P}$ is defined by

(1) If $r > 0$, then $r\vec{P}$ is the vector with same direction as $\vec{P}$ and magnitude r times the magnitude of $\vec{P}$.

(2) If $r < 0$, then $r\vec{P}$ is the vector with direction opposite to $\vec{P}$ and magnitude $|r|$ times the magnitude of $\vec{P}$.

(3) If $r = 0$, then $r\vec{P} = \vec{0}$.

(4) If $r = 1$, then $r\vec{P} = \vec{P}$.

When $r = -1$, $r\vec{P} = (-1)\vec{P}$ and we denote this vector by the symbol $-\vec{P}$. The vector $-P$ has the opposite sense of direction of P but has the same magnitude as shown in Figure 3-4.

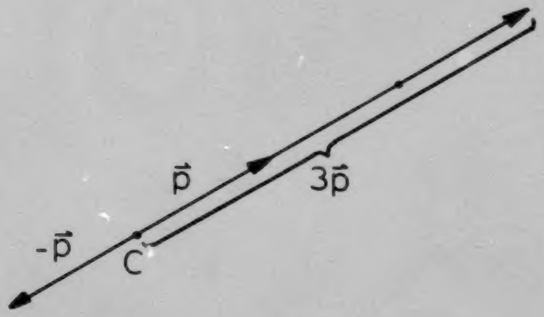


Figure 3-4

In accordance with our earlier definitions, we note that if $r \neq 0$, $r\vec{P}$ is always parallel to $\vec{P}$.

It is now possible to define one kind of division of two vectors.

DEFINITION. $\frac{\vec{A}}{\vec{B}} = k$, a scalar, if and only if $\vec{A} = k\vec{B}$; that is, if $\vec{A}$ and $\vec{B}$ are parallel.

We now can also make the following definition:

DEFINITION. $\vec{A} - \vec{B}$ means $\vec{A} + (-\vec{B})$. The quantity $\vec{A} - \vec{B}$ is called the difference of the two vectors $\vec{A}$ and $\vec{B}$.

Thus, in order to find the difference of two vectors, $\vec{A}$ and $\vec{B}$, we merely need to add the negative of the second to the first as shown in Figure 3-5.

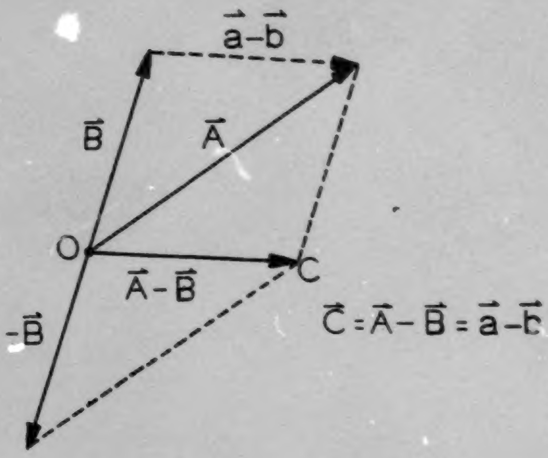


Figure 3-5

Figure 3-5 also shows that if $\vec{A} - \vec{B} = \vec{C}$, then $\vec{A} = \vec{B} + \vec{C}$.

Now that we have made the above definitions we are in a position to illustrate the distinction between the use of origin-vectors and other vectors referred to on p. 98. For example, the sum of vectors $\vec{a}$ and $\vec{b}$ in Figure 3-6 is equivalent to the sum of their respective origin-vectors $\vec{A}$ and $\vec{B}$.

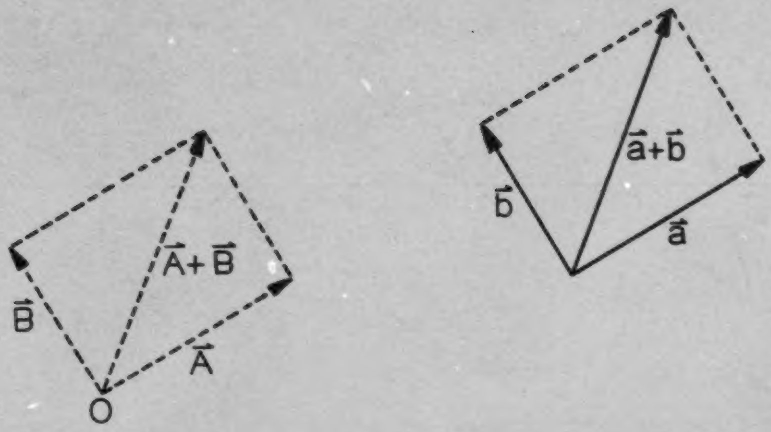

$\vec{A}+\vec{B}$
$\vec{B}$
$\vec{A}$
O
$\vec{a}+\vec{b}$
$\vec{b}$
$\vec{a}$

Figure 3-6

It is not even necessary that vectors $\vec{a}$ and $\vec{b}$ have the same initial point. (See Figure 3-7)

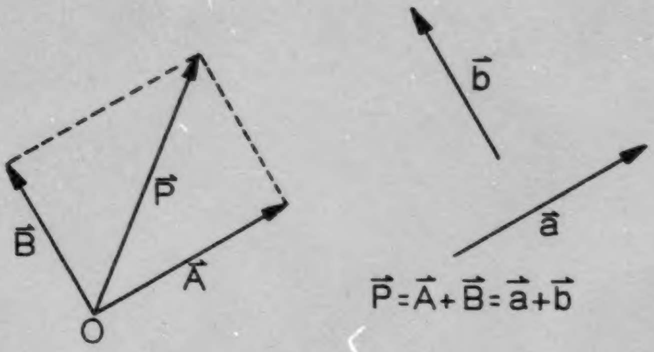

$\vec{P}$
$\vec{B}$
$\vec{A}$
O
$\vec{b}$
$\vec{a}$
$\vec{P}=\vec{A}+\vec{B}=\vec{a}+\vec{b}$

Figure 3-7

An important application of the above principle is shown in Figure 3-8 where the sum of $\vec{a}$ and $\vec{b}$ can be found by considering the equivalent of $\vec{b}$ with its initial point coincident with the terminal point of $\vec{a}$. This method can be applied to three or more vectors.

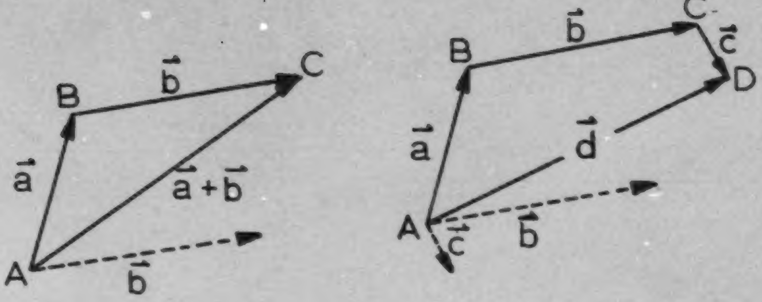
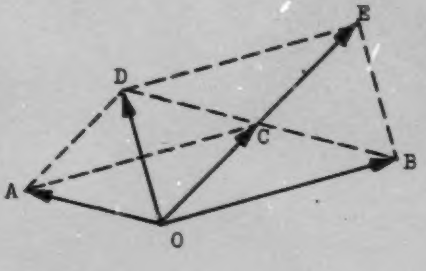

B
$\vec{b}$
C
$\vec{a}$
$\vec{a}+\vec{b}$
A
$\vec{b}$
C
$\vec{b}$
$\vec{c}$
B
D
$\vec{a}$
$\vec{d}$
A
$\vec{c}$
$\vec{b}$

Figure 3-8

In applying vector methods, physicists and other scientists often consider that they "move around a diagram", and then equate the corresponding vector sums. We could "move" from A to D directly, or from A through B and C to D . If the vector from A to D is called $\vec{d}$, then $\vec{d} = \vec{a} + \vec{b} + \vec{c}$. Likewise, one can go from A to C via two routes with the result that $\vec{a} + \vec{b} = \vec{d} - \vec{c}$.

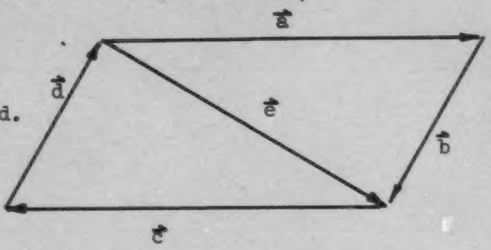
Exercises 3-3

1. Using the figure as given, supply the missing vector expressions.
   - (a) $\vec{A} + \vec{B} = ?$
   - (b) $\vec{D} - \vec{A} = ?$
   - (c) $\vec{A} + \vec{B} + \vec{C} = ?$
   - (d) $\vec{D} + \vec{B} = r\vec{C}$ (find r)
   - (e) $\vec{E} - ? = \vec{C}$



Quadrilaterals OCDA , OBCA , and OBED are parallelograms.

2. In the figure, A , B , C , and D are vertices of a parallelogram and determine the vectors indicated.



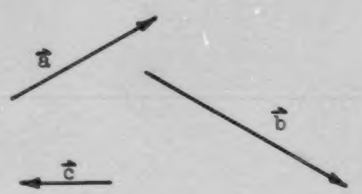
   (a) Express $\vec{c}$, $\vec{d}$, and $\vec{e}$ in terms of $\vec{a}$ and $\vec{b}$ alone.

   (b) Express $\vec{e}$ in terms of

      (i) $\vec{a}$ and $\vec{b}$
      (ii) $\vec{a}$ and $\vec{d}$
      (iii) $\vec{c}$ and $\vec{b}$
      (iv) $\vec{c}$ and $\vec{d}$

   (c) (i) What is the sum of $\vec{d}$, $\vec{e}$, and $\vec{c}$?
      (ii) What is the sum of $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$?

3. Draw on paper the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ as shown in the figure. Construct the vectors:



   (a) $\vec{b} + \vec{c}$
   (b) $\vec{a} - \vec{c}$
   (c) $\vec{c} - \vec{b}$
   (d) $\vec{a} + \vec{b} + \vec{c}$

4. By a drawing, show that if $\vec{a} + \vec{b} = \vec{c}$, then $\vec{b} = \vec{c} - \vec{a}$.

5. O , B , and X are collinear points. Find r such that

$$\vec{X} = r\vec{B}$$

   if

   (a) X is the midpoint of $\overline{OB}$.
   (b) B is the midpoint of $\overline{OX}$.
   (c) O is the midpoint of $\overline{BX}$.
   (d) X is $\frac{3}{4}$ of the way from O to B .
   (e) B is $\frac{2}{3}$ of the way from O to X .
   (f) O is $\frac{5}{8}$ of the way from B to X .

6. If $\vec{a} = \vec{b}$ and $\vec{c} = \vec{d}$, prove $\vec{a} + \vec{c} = \vec{b} + \vec{d}$.

7. If $|\vec{A}| = 3$, what is $|4\vec{A}|$? $|-5\vec{A}|$? $-|5\vec{A}|$?

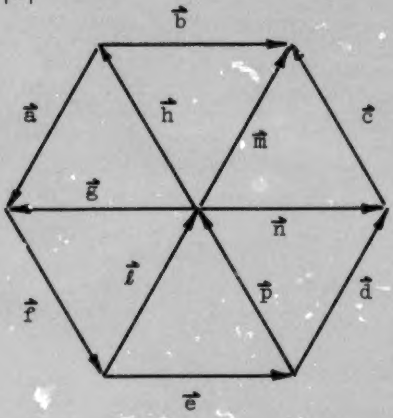
8. Prove: if $\vec{a} = \vec{b}$ and if r is a scalar, then $r\vec{a} = r\vec{b}$.

9. If $\vec{b}$ is a non-zero vector, and if $\frac{|\vec{a}|}{|\vec{b}|} = k$, what can you say about $|k\vec{b}|$?

10. The figure is a vector diagram based on a regular hexagon.

    (a) Write 6 vector equations which should occur to anyone in the class.

    (b) Write 6 more which are not obvious but which you could prove.



11. By using vectors, indicate 5 different paths in the plane by which one could move from $P = (1,2)$ to $Q = (4,6)$.

12. (a) If $|\vec{a}| = |\vec{b}|$, does $\vec{a} = \vec{b}$?

    (b) If $\vec{a} - \vec{b} = 0$, does $\vec{a} = \vec{b}$?

13. Prove $|\vec{a} + \vec{b}| \leqq |\vec{a}| + |\vec{b}|$

14. Letting 1 inch represent 2 miles, find graphically the resultant motion if a car travels 4 miles north and then 5 miles southeast, assuming the car travels in a plane.

15. Using the idea of resultant vectors and a scale of 1 inch to represent 2 miles per hour, solve the following problem graphically.

    A river has a 3 mile per hour current. A motor boat moves directly across the river (perpendicular to the current) at 5 miles per hour. How fast and in what direction would the boat be traveling if there were no current and the same power and heading were used in crossing the river?

16. Make a vector drawing with a scale of 1 inch to represent 10 pounds to solve the following problem.

    A body is acted on by two forces, $\vec{A}$ and $\vec{B}$, which make an angle of 70° with each other. The magnitude of $\vec{A}$ is 20 pounds and that of $\vec{B}$ is 30 pounds. What is the magnitude and direction of the resultant force?

17. Show that if $\vec{A}$ and $\vec{B}$ are distinct vectors, then $\vec{A} + (-1)\vec{B} = \vec{A} - \vec{B}$ lies on a line parallel to the line through the terminal points of $\vec{A}$ and $\vec{B}$, and similarly for $\vec{B} - \vec{A}$.

18. $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are consecutive vector sides of a quadrilateral. Prove that the figure is a parallelogram if and only if $\vec{b} + \vec{d} = \vec{0}$.

19. Prove that the sum of the six vectors drawn from the center of a regular hexagon to its vertices is the zero vector.

20. If we trace the perimeter of a polygon ABCD ... PA, and assign a vector $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ ..., $\vec{p}$ corresponding to each side as we traverse it, show that the vector sum $\vec{a} + \vec{b} + \vec{c} + \vec{d} + \ldots + \vec{p} = \vec{0}$. (It is this idea that physicists have in mind when they say, "The vector sum around a closed circuit is zero."

3-4. Properties of Vector Operations

We now derive several important algebraic properties of the operation of vector addition.

THEOREM 3-1. (Commutative Property)

$$\vec{P} + \vec{Q} = \vec{Q} + \vec{P}.$$

This follows from the definition of vector sum with the help of Figure 3-3.

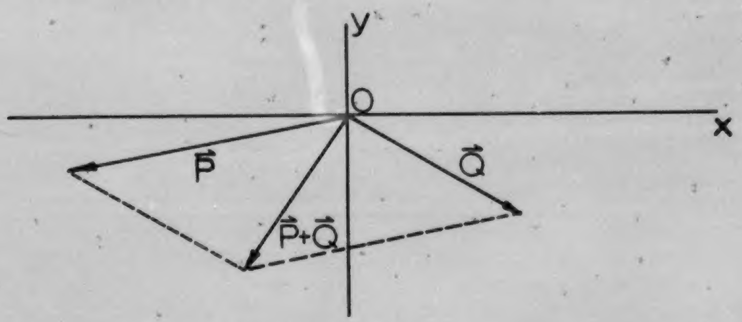


Figure 3-3

THEOREM 3-2. (Associative Property)

$$\vec{P} + (\vec{Q} + \vec{R}) = (\vec{P} + \vec{Q}) + \vec{R} .$$

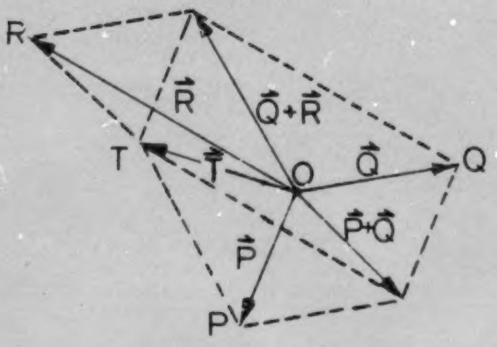


Figure 3-9

Figure 3-9 suggests a proof using the various parallelograms which appear. A much nicer proof will be given later.

THEOREM 3-3. (Additive Inverses)

For any vector $\vec{A}$, the equation

$$\vec{A} + \vec{X} = \vec{0}$$

is satisfied by $\vec{X} = (-1)\vec{A} = -\vec{A}$.

This follows immediately from the definition of addition of vectors and of $(-1)\vec{A}$.

Next we prove a theorem concerned with multiplying vectors by real numbers.

THEOREM 3-4. (Associative Property)

$$(rs)\vec{P} = r(s\vec{P}) .$$

This follows immediately from the definition of each member of the equation.

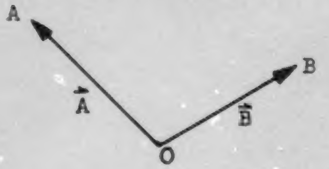
Exercises 3-4

1. By using the definition of subtraction, and the commutative and associative properties, show that

   (a) $\vec{B} + (\vec{A} - \vec{B}) = \vec{A}$

   (b) $(\vec{A} - \vec{B}) + \vec{B} = \vec{A}$

2. Draw on paper the figure showing $\vec{A}$ and $\vec{B}$. Locate point X such that $\vec{X} = p\vec{A} + q\vec{B}$,



   (a) if $p = 1$ and $q = 1$,

   (b) if $p = \frac{1}{3}$ and $q = \frac{2}{3}$,

   (c) if $p = 0$ and $q = \frac{1}{2}$,

   (d) if $p = \frac{1}{2}$ and $q = \frac{1}{2}$,

   (e) if $p = \frac{1}{4}$ and $q = \frac{5}{4}$.

   Can you make a conjecture about the values for p and q for which X is on $\overleftrightarrow{AB}$?

3. (a) Show by a vector drawing that the subtraction of vectors, e.g., $\vec{A} - \vec{B}$, is not commutative.

   (b) Is there a relation between the two differences, i.e., does $\vec{A} - \vec{B} = r(\vec{B} - \vec{A})$?

4. Prove Theorem 3-2.

5. Show that $-(\vec{P} + \vec{Q}) = -\vec{P} - \vec{Q}$.

6. Show that $(-r)\vec{P} = r(-\vec{P})$.

3-5. Characterization of the Points on a Line.

The term "linear combination" was first mentioned in Chapter 2 in connection with finding a point of division of a line segment. Now that we know how to add and subtract vectors and how to multiply a vector by a scalar, we can combine these operations to create other vectors, such as $2\vec{A} - 3\vec{B}$, $\frac{1}{2}(\vec{B} + \vec{C})$, and $(1 - x)\vec{A} + \frac{x}{2}\vec{B}$. To formalize this idea, we state the following definition:

DEFINITION. If $\vec{a}_1$, $\vec{a}_2$, ..., $\vec{a}_n$ are n vectors and x_1 , x_2 , ..., x_n are n scalars, the vector $x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n$ is said to be a linear combination of $\vec{a}_1$, $\vec{a}_2$, ..., $\vec{a}_n$.

In order to use vectors to prove theorems in geometry we need several basic theorems. The first one is concerned with expressing any vector in the plane as a linear combination of other vectors in the same plane.

THEOREM 3-5. If $\vec{a}$ and $\vec{b}$ are coplanar and non-parallel, then any third vector $\vec{c}$, which lies in the plane determined by $\vec{a}$ and $\vec{b}$, can be expressed as a unique linear combination of $\vec{a}$ and $\vec{b}$.

Given: Coplanar and non-parallel vectors $\vec{a}$ and $\vec{b}$, and $\vec{c}$ lying in their plane.

Prove: $\vec{c} = x\vec{a} + y\vec{b}$ where x and y are scalars.

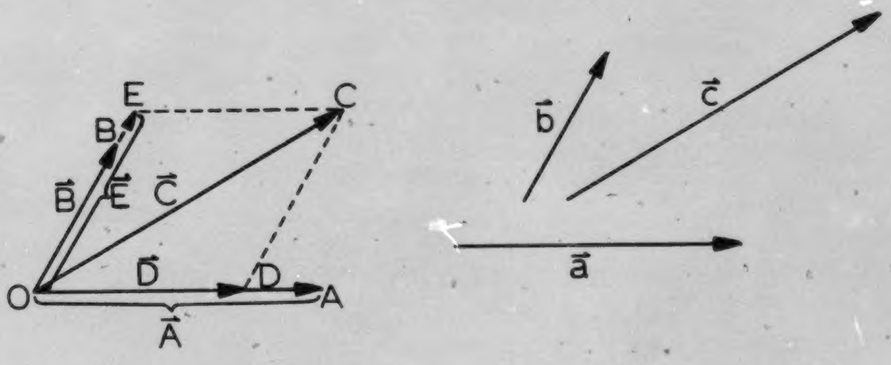


Figure 3-10

Inasmuch as vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ can be represented by their respective origin-vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ with terminal points A , B , and C as shown in Figure 3-10, we need only prove that $\vec{C} = x\vec{A} + y\vec{B}$. In this diagram we have chosen x and y positive although this restriction is not needed.

(1) Draw a line through C parallel to the line containing $\vec{B}$. Let D be the point of intersection of this line with the line containing $\vec{A}$.

(2) Since $\vec{D}$ is parallel to $\vec{A}$, it is some scalar multiple of $\vec{A}$. Thus, for some unique x, $\vec{D} = x\vec{A}$.

(3) Similarly, the vector $\vec{E}$, along the line containing $\vec{B}$, is a scalar multiple of $\vec{B}$. Thus, for some unique y, $\vec{E} = y\vec{B}$.

(4) Then $\vec{C} = \vec{D} + \vec{E} = x\vec{A} + y\vec{B}$ which shows $\vec{C}$ is a unique linear combination of $\vec{A}$ and $\vec{B}$. We have the equivalent statement: $\vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b}$.

We note that if $\vec{c}$ is parallel to $\vec{a}$ or $\vec{b}$, then $\vec{c}$ is a scalar multiple of either $\vec{a}$ or $\vec{b}$ alone.

THEOREM 3-6. (Distributive Properties)

1. $r(\vec{P} + \vec{Q}) = r\vec{P} + r\vec{Q}$.
2. $(r + s)\vec{P} = r\vec{P} + s\vec{P}$.

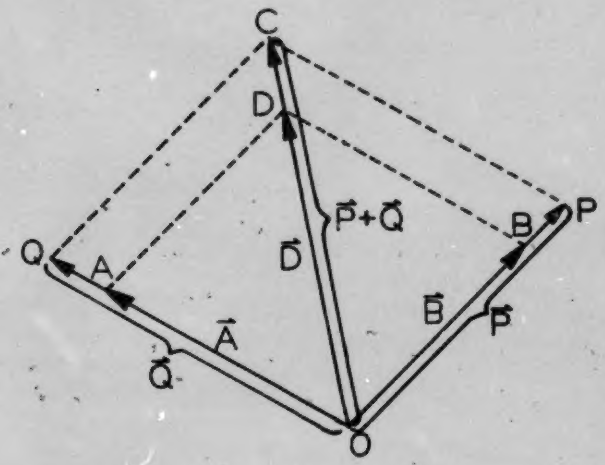


Figure 3-11

Proof of Part 1: $r(\vec{P} + \vec{Q}) = r\vec{P} + \vec{Q}$.

In this proof, we assume $\vec{P}$ and $\vec{Q}$ on distinct lines with $r > 0$.

(1) In Figure 3-11, $\vec{A} = r\vec{Q}$; $\vec{B} = r\vec{P}$.

Therefore: $|\vec{A}| = r|\vec{Q}|$, $|\vec{B}| = r|\vec{P}|$,

(2) $\dfrac{|\vec{B}|}{|\vec{A}|} = \dfrac{r|\vec{P}|}{r|\vec{Q}|} = \dfrac{|\vec{P}|}{|\vec{Q}|}$;

(3) $|\vec{B}| = d(O,B) = d(A,D)$,

$|\vec{A}| = d(O,A)$,

$|\vec{P}| = d(O,P) = d(Q,C)$,

$|\vec{Q}| = d(O,Q)$.

(4) Combining steps (2) and (3) we have $\dfrac{d(A,D)}{d(O,A)} = \dfrac{d(Q,C)}{d(O,Q)}$, and therefore $\triangle OAD \sim \triangle OQC$.

(5) $\therefore d(O,D) = r\,(d(O,C))$

$|\vec{D}| = r|\vec{C}| = |r\vec{C}|$.

(6) Since the vectors are in the same direction, we have $\vec{D} = r\vec{C}$.

(7) $\vec{D} = \vec{A} + \vec{B}$ or

$r\vec{C} = r\vec{Q} + r\vec{P}$, and since $\vec{C} = \vec{P} + \vec{Q}$,

$r(\vec{P} + \vec{Q}) = r\vec{P} + r\vec{Q}$.

Let us consider the special cases where the non-zero vectors $\vec{P}$ and $\vec{Q}$ are collinear. They are then parallel and have either the same or opposite senses.

If they have the same sense of direction, then

(1) By definition, $\vec{P} + \vec{Q}$ has the same sense of direction as $\vec{P}$ and $\vec{Q}$, and has magnitude $|\vec{P}| + |\vec{Q}|$.

(2) If $r > 0$, then $r(\vec{P} + \vec{Q})$ also has the same sense of direction as $\vec{P} + \vec{Q}$, $\vec{P}$, and $\vec{Q}$, and has magnitude $r(|\vec{P}| + |\vec{Q}|) = r|\vec{P}| + r|\vec{Q}|$ by definition and the distributive law.

(3) In the same way, since $r > 0$, $r\vec{P} + r\vec{Q}$ has the same sense of direction as $r\vec{P}$, $r\vec{Q}$, $\vec{P}$, and $\vec{Q}$, and has magnitude $|r\vec{P}| + |r\vec{Q}| = r|\vec{P}| + r|\vec{Q}|$.

(4) Since the vectors $r(\vec{P} + \vec{Q})$ and $r\vec{P} + r\vec{Q}$ have the same magnitude and the same sense of direction, they are equal, as was to be shown.

The case in which $\vec{P}$ and $\vec{Q}$ have opposite direction is treated in a similar fashion and the proof is left for class discussion.

The proof of the cases where $r \leq 0$ is also left for class discussion. The proof of the second part of the distributive law: $(r + s)\vec{P} = r\vec{P} + s\vec{P}$ is left as an exercise.

THEOREM 3-7. If $\vec{A}$ and $\vec{B}$ are distinct vectors not lying in the same line, then the vector $p\vec{A} + q\vec{B}$ will terminate on the line determined by the terminal points of $\vec{A}$ and $\vec{B}$ if and only if $p + q = 1$.

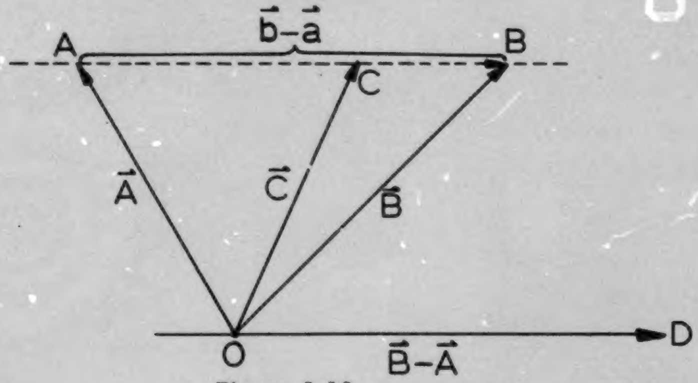


Figure 3-12

Proof:

(1) C is collinear with A and B if and only if $C = A$ or $\overline{AC} \parallel \overline{OD}$.

(2) $\overline{AC} \parallel \overline{OD}$ if and only if there exists a $q \neq 0$ such that

$$\vec{C} - \vec{A} = q(\vec{B} - \vec{A})$$

or

$$\vec{C} = \vec{A} + q\vec{B} - q\vec{A}$$

or

$$\vec{C} = q\vec{B} + (1 - q)\vec{A}$$

or

$$\vec{C} = p\vec{A} + q\vec{B} \qquad \text{where } p + q = 1 .$$

We note that if $q = 0$, then $\vec{C} = \vec{A}$.

The statement $\vec{C} = q\vec{B} + (1 - q)\vec{A}$ is a vector form of an equation of the line through A and B .

Each particular choice of p (and consequently of q) referred to in the Theorem 3-7 determines a vector to a point on the line $\overleftrightarrow{AB}$ in Figure 3-12. We can therefore describe subsets of the line $\overleftrightarrow{AB}$ by placing conditions on the scalars p and q .

The line $\overleftrightarrow{AB} = \{X : \vec{X} = p\vec{A} + q\vec{B} \text{ where } p + q = 1\}$
The segment $\overline{AB} = \{X : \vec{X} = p\vec{A} + q\vec{B} \text{ where } p + q = 1 \text{, and } p \geq 0 , q \geq 0\}$
The ray $\overrightarrow{AB} = \{X : \vec{X} = p\vec{A} + q\vec{B} \text{ where } p + q = 1 \text{ and } q \geq 0\}$
The ray $\overrightarrow{BA} = \{X : \vec{X} = p\vec{A} + q\vec{B} \text{ where } p + q = 1 \text{ and } p \geq 0\}$

The ray opposite to $\overrightarrow{AB} = \{X : \vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1$ and $q \leq 0\}$
The interior of $\overline{AB} = \{X : \vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1$ and $p > 0 , q > 0\}$.

Furthermore,

(i) if $\vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1$, $p > 0$ and $q > 0$, then X is an interior point of $\overline{AB}$,

(ii) if $\vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1$ and either p or q is zero, then X is an endpoint of $\overline{AB}$, and

(iii) if $\vec{X} = p\vec{A} + q\vec{B}$ where $p + q = 1$ and either $p < 0$ or $q < 0$, then X is a point of the line exterior to $\overline{AB}$.

We observe that in the vector representation $p\vec{A} + (1 - p)\vec{B}$ the scalar is also a coordinate in one of the coordinate systems for the line. When $p = 0$, we obtain $\vec{B}$; when $p = 1$, we obtain $\vec{A}$. The value of p which determines a vector $\vec{X}$ in this vector representation of the line $\overleftrightarrow{AB}$ is also the coordinate of the point X in the coordinate system for the line with origin B and unit-point A .

THEOREM 3-8. If P divides $\overline{AB}$ in the ratio n:m , then

$\vec{P} = \dfrac{m\vec{A} + n\vec{B}}{m + n}$ where $\vec{A}$, $\vec{B}$, and $\vec{P}$ are origin-vectors to points A , B , P respectively.

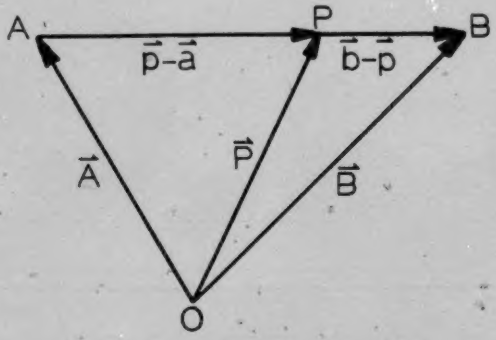


Figure 3-13

(1) Referring to Figure 3-13, $\dfrac{|\vec{p} - \vec{a}|}{|\vec{b} - \vec{p}|} = \dfrac{n}{m}$ (Given).

(2) $\dfrac{\vec{p} - \vec{a}}{\vec{b} - \vec{p}} = \dfrac{n}{m}$ (the vectors lie on the same line).

(3) $m(\vec{p} - \vec{a}) = n(\vec{b} - \vec{p})$.

(4) $m\vec{p} - m\vec{a} = n\vec{b} - n\vec{p}$.

(5) $m\vec{p} + n\vec{p} = m\vec{a} + n\vec{b}$.

(6) $(m + n)\vec{p} = m\vec{a} + n\vec{b}$, or $\vec{p} = \dfrac{m\vec{a} + n\vec{b}}{m + n} = \dfrac{m}{m + n}\vec{a} + \dfrac{n}{m + n}\vec{b}$.

(7) In terms of origin-vectors, we may then write:

$$\vec{P} = \frac{m\vec{A} + n\vec{B}}{m + n} = \frac{m}{m + n}\vec{A} + \frac{n}{m + n}\vec{B} .$$

Note: If P is the midpoint, then $\vec{P} = \frac{1}{2}(\vec{A} + \vec{B})$.

Exercises 3-5

1. Given vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ with their terminal points A , B , and C on a straight line, so that $\vec{C} = p\vec{A} + q\vec{B}$, $p + q = 1$.

   (a) What happens if $\vec{A}$ or $\vec{B}$ is the zero vector?

   (b) What are p and q if $\vec{C} = \vec{A}$?

   (c) What can we say about $\vec{C}$ if

      (i) $p > 0$ and $q > 0$?

      (ii) $p < 0$?

      (iii) $p = 0$?

   (d) Construct figures to illustrate the cases:

      (i) $p = q = \frac{1}{2}$

      (ii) $p = \frac{1}{3}$, $q = \frac{2}{3}$

      (iii) $p = -\frac{1}{4}$, $q = \frac{5}{4}$

      (iv) $p = \frac{3}{2}$, $q = -\frac{1}{2}$

2. (a) If the ratio of the division of a line segment is given by $n:m = 2:3$, find n and m so that $n + m = 1$.

   (b) Same as part (a) for $m:n = 5:-3$

3. Make a vector drawing to illustrate Theorem 3-5 when

   (a) $x = 2$, $y = 3$

   (b) $x = -2$, $y = 4$.

4. Prove Theorem 3-6, Part 2.

3-6. Components

We have used extensively the correspondence between points in the plane and vectors. It is fruitful to describe this correspondence in another way using the rectangular coordinates of a point. To each ordered pair of real numbers (a,b), there corresponds a unique vector emanating from O and terminating in that point and thus we make the following definition.

> **DEFINITION.** The symbol $[a,b]$ denotes the origin-vector to point (a,b). The number a is called the x-component of the vector and the number b, the y-component of the vector.

We now describe the operations involving vectors in terms of components.

THEOREM 3-9. If $\vec{X} = [a,b]$ and $\vec{Y} = [c,d]$,

$$\vec{X} + \vec{Y} = [a + c , b + d] .$$

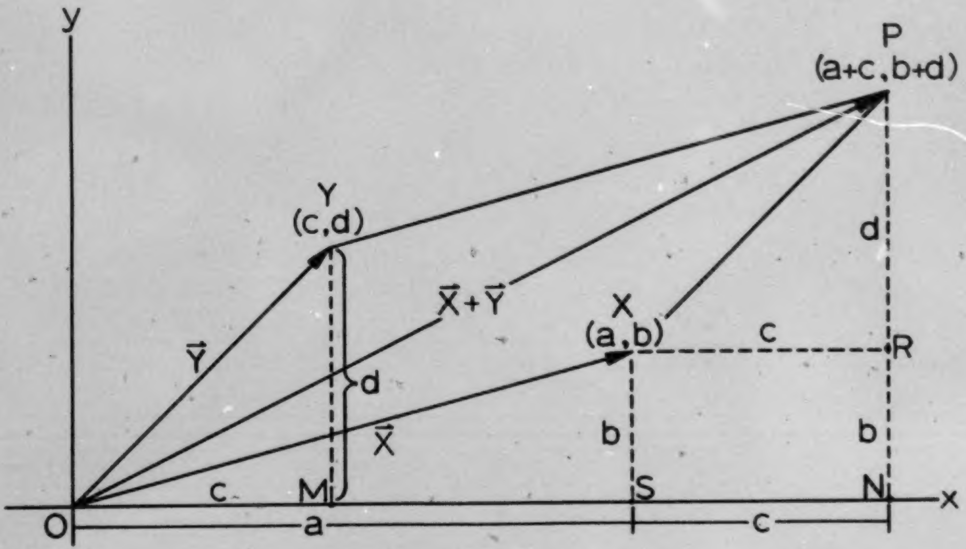


Figure 3-14

Proof. The parallelogram in Figure 3-14 is constructed according to the definition of addition of vectors.

Since $\triangle OMY \cong \triangle XRP$, $d(O,M) = d(X,R) = d(S,N) = c$ and $d(M,Y) = d(R,P) = d$. The vertex P opposite O is the point $(a + c, b + d)$, and this vertex is the terminal point of $\vec{X} + \vec{Y}$. If the vectors have the same or opposite directions, the proof follows immediately from the definition of vector addition. If $\vec{Y}$ is the zero vector $[0,0]$, then

$$[a,b] + [0,0] = X + Y = X = [a,b] = [a + 0, b + 0] .$$

THEOREM 3-10. If $\vec{X} = [a,b]$ and r is a real number, then $r\vec{X} = [ra,rb]$.

The proof is left as an exercise.

THEOREM 3-11. We prove, using components, a theorem learned earlier: Two non-zero vectors $\vec{X}$ and $\vec{Y}$ lie in the same line through the origin, if and only if $\vec{X} = r\vec{Y}$ for some real number r.

Proof. If $\vec{Y} = [a,b]$ and $\vec{X} = [ra,rb]$, then $\vec{X}$ and $\vec{Y}$ lie in the line $ay = bx$. Conversely, if $\vec{Y} = [a,b]$ and if $\vec{X}$ lies in the line which contains $\vec{Y}$, then the components of $\vec{X}$ must satisfy the equation $ay = bx$. Hence $\vec{X} = [ra,rb]$ for some real number r.

The vector $[1,0]$ is indicated by the letter i and $[0,1]$ by j. The i and j vectors could be written as $\vec{i}$ and $\vec{j}$ but, in accordance with common usage, we shall use the simpler notation. They represent the unit vectors along the horizontal and vertical axes respectively.

If $A = (a_1,a_2)$, the origin-vector $\vec{A}$ may be written as follows:

$$\vec{A} = [a_1,a_2] = [a_1,0] + [0,a_2] = a_1[1,0] + a_2[0,1] = a_1 i + a_2 j .$$

Note that a_1 and a_2 are the components of $\vec{A}$; $a_1 i$ and $a_2 j$ are called the component vectors of $\vec{A}$. We observe in Figure 3-15 that any origin-vector can be written uniquely as the sum of its component vectors. The magnitude of $\vec{A}$ is $\sqrt{a_1^2 + a_2^2}$.

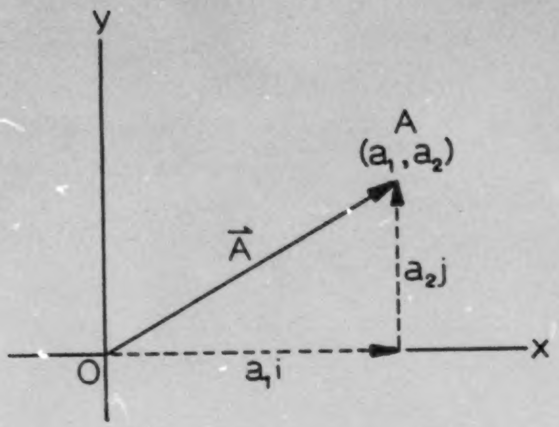


Figure 3-15

The use of components leads to a simple arithmetic of vectors, as will be seen in the following sections.

Example 1. Given $\vec{X} = [2,3]$ and $\vec{Y} = [-1,5]$,

Find $\vec{Z} = [4,-2]$ in terms of $\vec{X}$ and $\vec{Y}$.

We must find scalars r and s such that $\vec{Z} = r[2,3] + s[-1,5]$. Hence $[4,-2] = [2r,3r] + [-s,5s] = [2r - s , 3r + 5s]$.

Since the components of a given origin-vector are unique, we have:

$$2r - s = 4$$
$$3r + 5s = -2$$

We find that $r = \frac{18}{13}$, $s = \frac{-16}{13}$; hence $z = \frac{18}{13}[2,3] - \frac{16}{13}[-1,5]$.

We can form vector descriptions of lines and their subsets using components.

Example 2. Find the vector representation, in terms of a single parameter, for $\overleftrightarrow{AB}$ where $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$.

Solution. Let $\vec{P}$ be the origin-vector to any point P on $\overleftrightarrow{AB}$.

(1) $\vec{P} = r\vec{A} + (1-r)\vec{B}$ (Theorem 3-7)

$= r[3,4] + (1 - r)[-2,3]$

$= [3r,4r] + [-2 + 2r , 3 - 3r]$

(2) Thus $\overleftrightarrow{AB} = \{P:\vec{P} = [-2 + 5r , 3 + r]\}$.

Example 3. Find, using components, a vector representation of $\overline{AB}$ where $A = (3,4)$ and $B = (-2,3)$.

Solution. $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$. As in Example 2, any point P on $\overline{AB}$ can be represented by

$$\overline{AB} = \{P : \vec{P} = r\vec{A} + (1 - r)\vec{B}\} .$$

However we must place a restriction on r so that P will lie only on $\overline{AB}$. This condition will be met if $0 \leq r \leq 1$ since $P = A$ when $r = 1$ and $P = B$ when $r = 0$.

The complete solution is:

$$\overline{AB} = \{P : \vec{P} = [-2 + 5r , 3 + r] , 0 \leq r \leq 1\} .$$

Example 4. Find, using components, a vector representation of $\overrightarrow{BA}$ where $A = (3,4)$ and $B = (-2,3)$.

Solution. This problem differs from Example 3 in only one respect. We must now place a restriction on r so that P will lie only on $\overrightarrow{BA}$. This condition will be met if $r \geq 0$ since $P = B$ when $r = 0$ and P lies on the ray emanating from B and containing A when $r > 0$. The complete solution is:

$$\overrightarrow{AB} = \{P : \vec{P} = [-2 + 5r , 3 + r] , r \geq 0\} .$$

Example 5. Find the vector representation of the trisection points of $\overline{AB}$ where $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$.

Solution. Referring to Theorem 3-8, we have

$$\vec{P} = \frac{m\vec{A} + n\vec{B}}{m + n}$$

where P divides the segment in the ratio $n:m$.

There are two points of trisection, one where $n:m = 1:2$; the other where $n:m = 2:1$. We shall do the first part.

$$\vec{P} = \frac{2[3,4] + 1[-2,3]}{3} = \frac{2}{3}[3,4] + \frac{1}{3}[-2,3] = [\frac{4}{3}, \frac{11}{3}] .$$

Exercises 3-6

1. Find the components of

   (a) $[3,2] + [4,1]$.

   (b) $[3,-2] + [-4,1]$.

   (c) $4[5,6]$.

   (d) $-4[5,6]$.

   (e) $-1[5,6]$.

   (f) $-[5,6]$.

   (g) $3[4,1] + 2[-1,3]$.

   (h) $3[4,1] - 2[-1,3]$.

2. If $\vec{A} = [3,-5]$, $\vec{B} = [-1,6]$, $\vec{C} = [2,3]$, find the components of

   (a) $2\vec{A} + 3\vec{B} - \vec{C}$.

   (b) $\vec{A} - 2\vec{B} + 3\vec{C}$.

   (c) $2(\vec{A} + \vec{B}) - 3(\vec{B} - \vec{C})$.

   (d) $5(\vec{A} - \vec{C}) + 3(\vec{C} - \vec{A})$.

   (e) $3(\vec{A} + \vec{B} - \vec{C}) + 2(\vec{A} - \vec{B} + \vec{C})$.

   (f) $5(\vec{C} - \vec{A} + \vec{B}) - 3(\vec{B} + \vec{A} - \vec{C})$.

3. What is the x component of i ? of j ?

4. Find the magnitude of the following vectors:

   (a) $i + j$.

   (b) $3i - 4j$.

   (c) $ai + bj$.

   (d) $(\cos \theta)i + (\sin \theta)j$.

5. Vector $\vec{F}$ is drawn from $A = (4,2)$ to $B = (5,-1)$. Write its origin-vector $\vec{F}$ in terms of i and j .

6. Express the zero vector $\vec{0}$ in terms of two distinct non-collinear vectors $\vec{X}$ and $\vec{Y}$ lying in the same plane.

7. In terms of i and j , describe the vector represented by the arrow extending from 0 to the midpoint of the segment joining $(2,5)$ and $(5,8)$.

8. In terms of i and j , describe

   (a) the unit vector making an angle of 30° with the x-axis.

   (b) the unit vector making an angle of -30° with the x-axis.

   (c) the unit vector having the same direction as $4i - 3j$.

9. Find x and y so that

   (a) $x[3,-1] + y[3,1] = [5,6]$.

   (b) $x[3,2] + y[2,3] = [1,2]$.

   (c) $x[3,2] + y[-2,3] = [5,6]$.

   (d) $x[3,2] + y[6,4] = [-3,-2]$ (infinitely many solutions. Why?)

10. Represent an arbitrary vector $[a,b]$ as a linear combination of

    (a) $[1,0]$ and $[0,1]$.

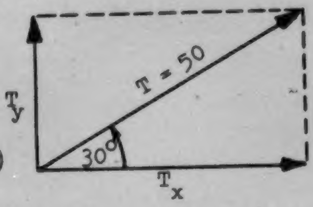
    (b) $[1,1]$ and $[-1,1]$.

    (c) $[-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]$ and $[-1,0]$.

11. Physical forces possess both magnitude and direction and therefore may be represented by vectors. In physics problems it is often convenient to use x-components and y-components to represent the horizontal and vertical components of a force.

    Suppose a sled is being pulled along level ground by a cord making an angle of 30° with the ground. The tension (magnitude of the pulling force) in the cord is 50 pounds. What is the component of the force parallel to the ground, and what is the component of the force perpendicular to the ground?

    (Hint: With the force vector emanating from the origin, the horizontal vector will be $[T \cos 30^\circ, 0]$ and the vertical vector will be $[0, T \sin 30^\circ]$.)



12. Two forces act simultaneously at the same point. The first has a magnitude of 20 pounds, and direction 37° above the horizontal and toward the right. The other force has a magnitude of 30 pounds and direction 30° below the horizontal and toward the right. Find the vector which represents the resultant of these two forces.

13. Refer to the forces of Exercise 12.

    (a) At what angle must the second force act if the resultant acts horizontally toward the right?

    (b) At what angle must the second force act if the resultant acts vertically?

14. Suppose three forces act simultaneously at the same point. (It can be seen from the commutative and associative properties of addition for vectors that there is but one resultant for all three, no matter which two are taken first.) Find the resultant of these three forces: 20 pounds acting due west, 30 pounds acting northwest, and 40 pounds acting due south.

15. If two forces have the same magnitude but act in opposite directions, they are said to be in equilibrium and each is called the equilibrant of the other.

    (a) Find the magnitude and direction of the equilibrant of the resultant of two forces, one pulling due north with a magnitude of 20 pounds and the other pulling southeast with a magnitude of 30 pounds.

    (b) If a third force of 10 pounds acting due east is added, find the force which will provide equilibrium for the whole system.

16. A picture weighing ten pounds is suspended evenly by a wire going over a hook on the wall. If the two ends of the wire make an angle of 140° at the hook, find the tension in the wire. (See Exercise 11 for the use of "tension".)

17. Prove Theorems 3-1, 3-2, and 3-6 using components.

18. Prove Theorem 3-10.

19. Find vector representations, in terms of a single parameter for the sets described below:

    (a) $\overleftrightarrow{AB}$ where $\vec{A} = [2,3]$ and $\vec{B} = [-4,5]$
    (b) $\overleftrightarrow{AB}$ where $\vec{A} = [1,3]$ and $\vec{B} = [3,9]$
    (c) $\overleftrightarrow{AB}$ where $\vec{A} = [4,-7]$ and $\vec{B} = [4,2]$
    (d) $\overleftrightarrow{AB}$ where $\vec{A} = [2]$ and $\vec{B} = [3]$
    (e) $\overline{AB}$ where $\vec{A} = [-3,2]$ and $\vec{B} = [1,-2]$
    (f) $\overline{AB}$ where $\vec{A} = [1]$ and $\vec{B} = [2]$
    (g) $\overline{AB}$ where $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$
    (h) $\overrightarrow{AB}$ where $\vec{A} = [1,-2]$ and $\vec{B} = [-3,2]$
    (i) $\overrightarrow{AB}$ where $\vec{A} = [2]$ and $\vec{B} = [1]$
    (j) $\overrightarrow{AB}$ where $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$
    (k) $\overline{BA}$ where $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$
    (l) $\overrightarrow{BA}$ where $\vec{A} = [1]$ and $\vec{B} = [2]$
    (m) The ray opposite to $\overrightarrow{AB}$ where $\vec{A} = [3,4]$ and $\vec{B} = [-2,3]$
    (n) The interior of segment $\overline{AB}$ where $\vec{A} = [-3,2]$ and $\vec{B} = [1,-2]$

20. Find the vector representations of the midpoints and trisection points of the following line segments:

(a) $\overline{AB}$ where $A = [0,0]$ and $B = [6,12]$

(b) $\overline{AB}$ where $A = [-3,2]$ and $B = [10,-11]$

(c) $\overline{AB}$ where $A = [a_1,a_2]$ and $B = [b_1,b_2]$

21. Find the vector representations of the points which divide the directed segment (P,Q) in the ratio $\frac{r}{s}$ where:

(a) $P = [4,6]$, $Q = [-1,11]$, and $\frac{r}{s} = \frac{2}{3}$

(b) $P = [4]$, $Q = [11]$, and $\frac{r}{s} = \frac{3}{4}$

(c) $P = [-3,-2]$, $Q = [3,2]$, and $\frac{r}{s} = 1$

(d) $P = [-1,4]$, $Q = [9,-5]$, and $\frac{r}{s} = \frac{1}{5}$

(e) $P = [\frac{3}{2},\frac{2}{3}]$, $Q = [\frac{1}{13},\frac{8}{13}]$, and $\frac{r}{s} = \frac{\sqrt{2}}{\pi}$

(f) $P = [4]$, $Q = [11]$, and $\frac{r}{s} = \frac{6}{8}$

3-7. Inner Product.

Our algebra of vectors does not yet include multiplication of one vector by another. In order to make a definition which will have significant consequences, we investigate the angle between two vectors.

> **DEFINITION.** Let $\vec{X}$ and $\vec{Y}$ be any two non-zero vectors. Then by the angle between $\vec{X}$ and $\vec{Y}$ we mean the angle whose sides contain $\vec{X}$ and $\vec{Y}$. This angle has a unique degree measure between 0° and 180° (inclusive).

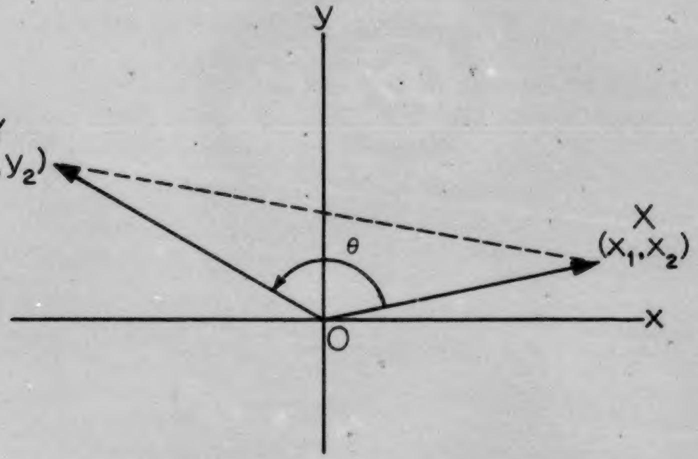


Figure 3-16

Let θ denote the angle between $\vec{X}$ and $\vec{Y}$. The law of cosines, applied to triangle OXY, enables us to write

$$\left(d(X,Y)\right)^2 = |\vec{X}|^2 + |\vec{Y}|^2 - 2|\vec{X}|\,|\vec{Y}| \cos \theta .$$

The term $|X|\,|Y| \cos \theta$ has significant physical applications which lead us to a useful vector concept. One such application deals with the work done in applying a force through a given distance. Since we must consider the direction and magnitude of both the force which is applied and the motion which takes place, it is customary to represent them by vectors $\vec{F}$ and $\vec{S}$, where $s = |\vec{S}|$ is the distance.

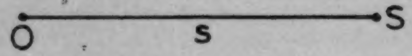


Figure 3-17

In Figure 3-17, an object at O is moved a distance s by a force $\vec{F}$. This force is applied to the object along a straight line and in the same direction as that line so that all of the force acts in the direction of motion.

On the other hand, if the force is applied at an angle θ, as shown in Figure 3-18, only that vector component of the force, $\vec{F}_x$, which produces the motion is effective in performing the work done.

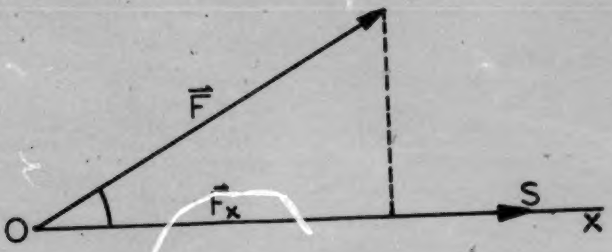


Figure 3-18

In Figure 3-18, $d(O,S) = s = |\vec{S}|$ so

$$\text{Work} = |\vec{F}_x|s = |\vec{F}|s \cos \theta = |\vec{F}||\vec{S}| \cos \theta .$$

DEFINITION. Let $\vec{X}$ and $\vec{Y}$ be any non-zero vectors. Then the inner product, $\vec{X} \cdot \vec{Y}$, of the two vectors is the real number

$$|\vec{X}||\vec{Y}| \cos \theta$$

where $|\vec{X}|$ is the magnitude of $\vec{X}$, $|\vec{Y}|$ is the magnitude of $\vec{Y}$, and θ is the angle between $\vec{X}$ and $\vec{Y}$. If either $\vec{X}$ or $\vec{Y}$ is the zero vector, $\vec{X} \cdot \vec{Y}$ is defined to be zero.

The inner product $\vec{X} \cdot \vec{Y}$ is usually read "vector X dot vector Y" and is therefore sometimes called the "dot product". Notice that the inner product is an operation that assigns to each pair of vectors a real number rather than a vector. The operation is obviously commutative.

In view of the above definition, Work = $\vec{F} \cdot \vec{S}$. Also $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, $\vec{a} \cdot \vec{b} = \vec{A} \cdot \vec{B}$.

Example. Evaluate $\vec{X} \cdot \vec{Y}$ if $|\vec{X}| = 2$, $|\vec{Y}| = 3$ and (a) $\theta = 0^\circ$, (b) $\theta = 45^\circ$, (c) $\theta = 90^\circ$, (d) $\theta = 180^\circ$.

Solution.

(a) $\vec{X} \cdot \vec{Y} = 2 \cdot 3 \cos 0^\circ = 2 \cdot 3 \cdot 1 = 6$

(b) $\vec{X} \cdot \vec{Y} = 2 \cdot 3 \cos 45^\circ = 2 \cdot 3 \cdot \frac{\sqrt{2}}{2} = 3\sqrt{2}$

(c) $\vec{X} \cdot \vec{Y} = 2 \cdot 3 \cos 90^\circ = 2 \cdot 3 \cdot 0 = 0$

(d) $\vec{X} \cdot \vec{Y} = 2 \cdot 3 \cos 180^\circ = 2 \cdot 3 \cdot (-1) = -6$

The inner product has many applications. One of these is a test for perpendicularity.

THEOREM 3-12. If $\vec{X}$ and $\vec{Y}$ are non-zero vectors, then they are perpendicular if and only if

$$\vec{X} \cdot \vec{Y} = 0 .$$

Proof. According to the definition of inner product

$$\vec{X} \cdot \vec{Y} = |\vec{X}| \cdot |\vec{Y}| \cos \theta .$$

This product of real numbers is zero if and only if one of its factors is zero. Since $\vec{X}$ and $\vec{Y}$ are non-zero vectors, the numbers $|\vec{X}|$ and $|\vec{Y}|$ are not zero. Therefore the product is zero if and only if $\cos \theta = 0$, which is the case if and only if $\vec{X}$ and $\vec{Y}$ are perpendicular.

The following theorem supplies a useful formula for the inner product of vectors.

THEOREM 3-13. If $\vec{X} = [x_1, x_2]$ and $\vec{Y} = [y_1, y_2]$, then

$$\vec{X} \cdot \vec{Y} = x_1 y_1 + x_2 y_2 .$$

Proof. From the law of cosines and the distance formula we can now write (see Figure 3-16)

$$\vec{X}\cdot\vec{Y} = |\vec{X}|\,|\vec{Y}|\cos\theta = \frac{|\vec{X}|^2 + |\vec{Y}|^2 - (d(X,Y))^2}{2}$$

$$= \frac{1}{2}[x_1^2 + x_2^2 + y_1^2 + y_2^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2]$$

$$= \frac{1}{2}(2x_1y_1 + 2x_2y_2) = x_1y_1 + x_2y_2\ .$$

Example 1. If $\vec{X} = [8,-6]$ and $\vec{Y} = [3,4]$, show that $\vec{X}$ and $\vec{Y}$ are perpendicular.

Solution. $\vec{X}\cdot\vec{Y} = 8\cdot 3 + (-6)\cdot 4 = 24 - 24 = 0$.
Since $\vec{X}$ and $\vec{Y}$ are non-zero vectors, Theorem 3-12 shows that they are perpendicular.

Example 2. Find the angle between the vectors $\vec{A} = [4,3]$ and $\vec{B} = [-2,2]$.

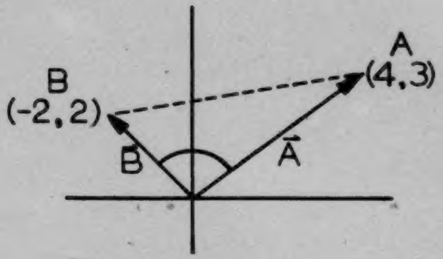


Figure 3-19

Solution.

$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta$$

$$\vec{A}\cdot\vec{B} = (4)(-2) + (3)(2) = -2$$

$$|\vec{A}| = 5\ ,\ |\vec{B}| = 2\sqrt{2}$$

$$\therefore \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{-2}{10\sqrt{2}} = -\frac{\sqrt{2}}{10} \approx -.141$$

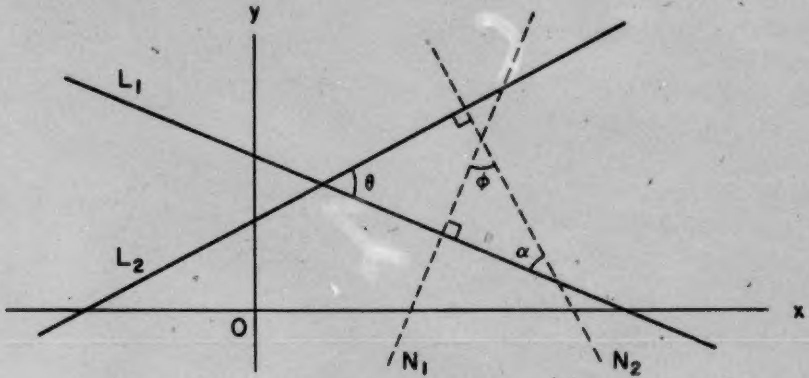
$$\theta \simeq 98^\circ$$

We shall find further application for the formula

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}\ .$$

The Angle Between Two Lines. An application of this formula can be made to find the angles formed by two lines with equations in rectangular form. Suppose the lines are L_1 and L_2 with respective equations

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.



In Chapter 2 we learned that the respective normals N_1 and N_2 have direction numbers (a_1,b_1) and (a_2,b_2) . We may take these as vector components of vectors along N_1 and N_2 . From the diagram, $\angle\theta$ and $\angle\Phi$ have equal measure since each is the complement of $\angle\alpha$; hence, we may find θ , the measure of the angle between L_1 and L_2 , by finding Φ , the measure of the angle between their normals. Therefore

$$\cos\theta = \cos\Phi = \frac{[a_1,b_1]\cdot[a_2,b_2]}{|[a_1,b_1]|\ |[a_2,b_2]|} = \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\ \sqrt{a_2^2 + b_2^2}}$$

This is the same formula we found in Chapter 2 by another approach.

Example. Find the angles formed by the lines with equations $3x + 4y + 5 = 0$ and $5x + 12y + 9 = 0$.

Solution. Direction numbers for the normals to these lines are $(3,4)$ and $(5,12)$; therefore,

$$\cos\theta = \frac{[3,4]\cdot[5,12]}{|[3,4]|\,|[5,12]|} = \frac{15 + 48}{\sqrt{3^2 + 4^2}\ \sqrt{5^2 + 12^2}} = \frac{63}{5\cdot 13} = \frac{63}{65},$$

$$\cos\theta \approx .969 , \qquad \text{and} \qquad \theta \approx 14^\circ .$$

The angles formed have measure 14° and 166° .

Exercises 3-7

1. If $i = [1,0]$ and $j = [0,1]$, find
   - (a) $i \cdot j$
   - (b) $j \cdot i$
   - (c) $i \cdot i$
   - (d) $j \cdot j$
   - (e) $(i + j) \cdot (i - j)$
   - (f) $(2i + 3j) \cdot (4i - 5j)$
   - (g) $(ai + bj) \cdot (ci + dj)$.

2. If $\vec{A} = [3,-5]$, $\vec{B} = [-2,1]$, $\vec{C} = [4,-3]$, find:
   - (a) $\vec{A} \cdot \vec{B}$
   - (b) $2\vec{A} \cdot 3\vec{B}$
   - (c) $3\vec{A} \cdot (\vec{B} + \vec{C})$
   - (d) $2\vec{B} \cdot (3\vec{A} + 2\vec{C})$
   - (e) $(\vec{A} + \vec{B}) \cdot (\vec{A} - \vec{B})$
   - (f) $(2\vec{B} + 3\vec{C}) \cdot (2\vec{B} - 3\vec{C})$
   - (g) $(3\vec{A} + 5\vec{B}) \cdot (3\vec{B} - 2\vec{C})$
   - (h) $(\vec{A} + \vec{B} - \vec{C}) \cdot (\vec{B} - \vec{A} + \vec{C})$
   - (i) $(2\vec{A} - 3\vec{B} + 4\vec{C}) \cdot (5\vec{A} - 2\vec{C} + 4\vec{B})$
   - (j) $\vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} + \vec{C} \cdot \vec{C}$

3. Find the angle between $\vec{X}$ and $\vec{Y}$ if $|\vec{X}| = 2$, $|\vec{Y}| = 3$ and $\vec{X} \cdot \vec{Y}$ is
   - (a) 0
   - (b) 1
   - (c) -2
   - (d) 3
   - (e) -4
   - (f) 5
   - (g) 6
   - (h) -6

4. Given
   - (a) $\vec{A} = 4i - 3j$, find $|\vec{A}|^2$.
   - (b) $\vec{B} = 12i + 5j$, find $|\vec{B}|^2$.

5. If $\vec{X} = 3i + 4j$, determine w so that $\vec{Y}$ is perpendicular to $\vec{X}$, if $\vec{Y}$ is
   - (a) $wi + 4j$
   - (b) $wi - 4j$
   - (c) $4i + wj$
   - (d) $wi - 3j$
   - (e) Find an origin-vector in component form which is perpendicular to $\vec{X}$ and four times as long. (two answers)

6. Given $\vec{A} = 2i - j$ and $\vec{B} = 3i + 6j$ as sides of $\triangle AOB$; what kind of a triangle is $\triangle AOB$? Find the third side $\vec{c}$ in terms of $\vec{A}$ and $\vec{B}$. Find $\vec{C}$, the origin-vector of $\vec{c}$, in terms of its unit vectors.

7. Let $\vec{A} = 2i - 3j$, $\vec{B} = -2i + j$. Find
   - (a) the angle between $\vec{A}$ and $\vec{B}$.
   - (b) the work done by $\vec{A}$, considered as a force vector, in moving a particle from the origin to $S = (2,0)$ along the x-axis.

8. A sled is pulled a distance of s ft. by a force of f lbs., where F represents the force which makes an angle of θ with the horizontal. Find the work done if

   (a) $s = 100$ ft., $f = 10$ lbs., $\theta = 20^\circ$.

   (b) $s = 1000$ ft., $f = 10$ lbs., $\theta = 30^\circ$.

9. In Problem (8), how far can the sled be dragged if the number of available foot pounds of work is 1000 and if

   (a) $f = 100$ lbs., $\theta = 20^\circ$.

   (b) $f = 100$ lbs., $\theta = 89^\circ$.

10. Let $\vec{A} = (\cos\theta)i + (\sin\theta)j$ and
    $\vec{B} = (\cos\Phi)i + (\sin\Phi)j$.

    Draw these vectors in the xy-plane.

    (a) Find $\vec{A}\cdot\vec{B}$, $|\vec{A}|$, $|\vec{B}|$

    (b) Use those results to prove that
    $\cos(\Phi - \theta) = \cos\Phi\cos\theta + \sin\Phi\sin\theta$.

11. Prove: $-1 \le \dfrac{\vec{X}\cdot\vec{Y}}{|\vec{X}|\,|\vec{Y}|} \le 1$.

12. Comment on the following: there is an associative law for vector addition: $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$. Therefore, there may be an associative law for inner products: $\vec{A}\cdot(\vec{B}\cdot\vec{C}) = (\vec{A}\cdot\vec{B})\cdot\vec{C}$

3-8. Laws and Applications of the Inner (Dot) Product.

A useful fact about inner products is that they have some of the algebraic properties of products of numbers. The following theorem gives two such properties.

THEOREM 3-14. If $\vec{X}$, $\vec{Y}$, $\vec{Z}$ are any vectors, then

(a) $\vec{X}\cdot(\vec{Y} + \vec{Z}) = \vec{X}\cdot\vec{Y} + \vec{X}\cdot\vec{Z}$

(b) $(t\vec{X})\cdot\vec{Y} = t(\vec{X}\cdot\vec{Y}) = (\vec{X})\cdot(t\vec{Y})$.

Part (b) states "a scalar multiple of a dot product can be attached to either vector factor."

Proof. Let $\vec{X} = [x_1, x_2]$, $\vec{Y} = [y_1, y_2]$, $\vec{Z} = [z_1, z_2]$. Then

(a)
$$\begin{aligned}\vec{X} \cdot (\vec{Y} + \vec{Z}) &= [x_1, x_2] \cdot [y_1 + z_1 , y_2 + z_2] \\ &= x_1(y_1 + z_1) + x_2(y_2 + z_2) \\ &= x_1y_1 + x_2y_2 + x_1z_1 + x_2z_2 \\ &= \vec{X} \cdot \vec{Y} + \vec{X} \cdot \vec{Z} ,\end{aligned}$$

(b)
$$\begin{aligned}(t\vec{X}) \cdot \vec{Y} &= [tx_1, tx_2] \cdot [y_1, y_2] \\ &= tx_1y_1 + tx_2y_2 \\ &= t(x_1y_1 + x_2y_2) \\ &= t(\vec{X} \cdot \vec{Y}) .\end{aligned}$$

Corollary. $\vec{X} \cdot (a\vec{Y} + b\vec{Z}) = a(\vec{X} \cdot \vec{Y}) + b(\vec{X} \cdot \vec{Z})$.

The proofs of this corollary and the last part of Theorem 3-14 are left as exercises.

We may now use the inner product to prove theorems in geometry which involve perpendicularity.

Example 1. Show that the diagonals of a rhombus are perpendicular.

Solution. Choose the origin as one vertex of the rhombus. The two adjacent sides can be represented by the vectors $\vec{A}$ and $\vec{B}$ with $|\vec{A}| = |\vec{B}|$.

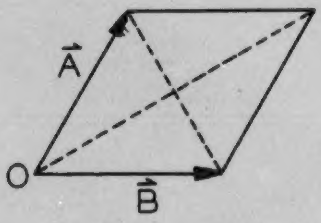


Figure 3-20

Thus one diagonal is represented by $\vec{A} + \vec{B}$ and the other diagonal is parallel to $\vec{A} - \vec{B}$. To test for perpendicularity we calculate the inner product of these two vectors, using Theorem 3-14.

$$\begin{aligned}(\vec{A}+\vec{B})\cdot(\vec{A}-\vec{B}) &= (\vec{A}+\vec{B})\cdot\vec{A} - (\vec{A}+\vec{B})\cdot\vec{B}\\ &= \vec{A}\cdot\vec{A}+\vec{B}\cdot\vec{A}-\vec{A}\cdot\vec{B}-\vec{B}\cdot\vec{B}\\ &= |\vec{A}|^2 - |\vec{B}|^2 .\end{aligned}$$

But $|\vec{A}| = |\vec{B}|$, so that the inner product is zero and hence the diagonals are perpendicular.

Example 2. Prove that the altitudes of a triangle are concurrent.

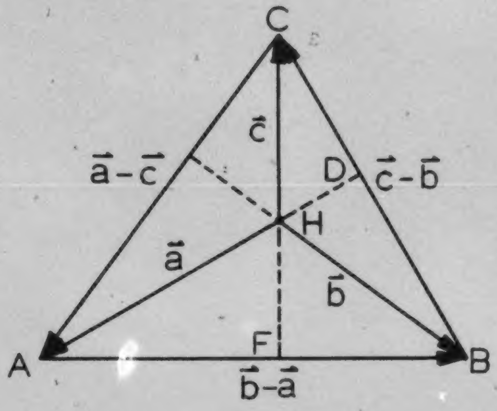


Figure 3-21

Proof. Refer to Figure 3-21: Let $\overline{BE}$ and $\overline{CF}$ be altitudes of $\triangle ABC$. Then $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ must intersect at some point H. AH intersects BC at some point D. We must prove $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$.

(1) $\vec{b}\cdot(\vec{a}-\vec{c}) = \vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{c} = 0$; (Why?)
thus $\vec{b}\cdot\vec{a} = \vec{b}\cdot\vec{c}$.

(2) Similarly, $\vec{c}\cdot(\vec{b}-\vec{a}) = \vec{c}\cdot\vec{b} - \vec{c}\cdot\vec{a} = 0$;
thus $\vec{c}\cdot\vec{a} = \vec{c}\cdot\vec{b}$.

(3) $\vec{b}\cdot\vec{a} = \vec{c}\cdot\vec{a}$. (Why?)

(4) $\vec{c}\cdot\vec{a} - \vec{b}\cdot\vec{a} = 0$.

(5) $(\vec{c}-\vec{b})\cdot\vec{a} = 0$ and $\vec{a} \perp (\vec{c}-\vec{b})$.

(6) Hence $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$ and the three altitudes are concurrent.

The inner product can be used to derive another result. Let $\vec{X} = [x_1, x_2]$ be a non-zero vector. Then $\vec{X'} = [-x_2, x_1]$ is also a non-zero vector and we have

$$\vec{X}\cdot\vec{X'} = [x_1,x_2]\cdot[-x_2,x_1] = -x_1x_2 + x_1x_2 = 0 .$$

Hence by Theorem 3-12, $\vec{X}$ and $\vec{X'}$ are perpendicular and the angle between the vectors is 90° . Now let $\vec{Y} = [y_1,y_2]$ be any non-zero vector. We now calculate $\vec{X'}\cdot\vec{Y}$.

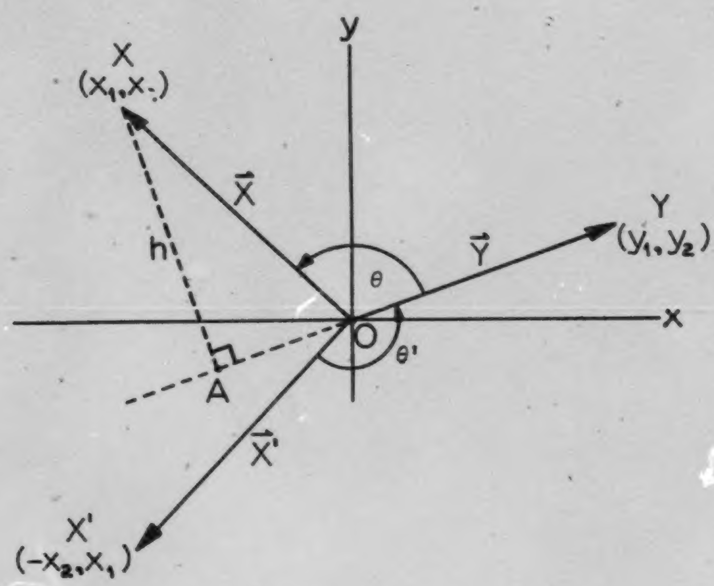


Figure 3-22

To do so we must determine the angle between the vectors $\vec{X'}$ and $\vec{Y}$. The relationship of this angle to angle θ is not always the same. In Figure 3-22 the angle θ' between $\vec{X'}$ and $\vec{Y}$ is $360^\circ - (90^\circ + \theta)$. If $\vec{Y}$ were near the positive side of the y-axis, the angle θ would be $90^\circ + \theta$. If $\vec{Y}$ were between $\vec{X}$ and $\vec{X'}$, the angle θ' would be $90^\circ - \theta$. If $\vec{Y}$ were near the negative side of the y-axis, the angle would be $\theta - 90^\circ$. Therefore, we have

$$\cos\theta' = \left\{\begin{array}{l} \cos[360^\circ - (90^\circ + \theta)] , \\ \cos(90^\circ + \theta) , \\ \cos(90^\circ - \theta) , \\ \text{or } \cos(\theta - 90^\circ) , \end{array}\right\} = \pm \sin\theta .$$

Therefore, in any case, since $X' = [-x_2,x_1]$,

$$\vec{X'}\cdot\vec{Y} = [-x_2,x_1]\cdot[y_1,y_2] = x_1y_2 - x_2y_1 = |\vec{X}||\vec{Y}|\cos\theta' = \pm|\vec{X}||\vec{Y}|\sin\theta .$$

But from the figure, we see that $|\overrightarrow{X}| \sin\theta$ is the length of the altitude h drawn from X to line $\overleftrightarrow{OY}$ in $\triangle OXY$. Thus the area K of $\triangle OXY$ is given by

$$K = \frac{1}{2}|\overrightarrow{Y}|h .$$

However, since $h = |\overrightarrow{X}| \sin\theta$,

$$K = \frac{1}{2}|\overrightarrow{Y}|\,|\overrightarrow{X}| \sin\theta = \frac{1}{2}|x_1y_2 - x_2y_1| .$$

3-9. Resolution of Vectors.

In the first discussion on vector components (Section 3-6), it was noted that the vector $\overrightarrow{X} = [a,b]$ had a as its x-component and b as its y-component.

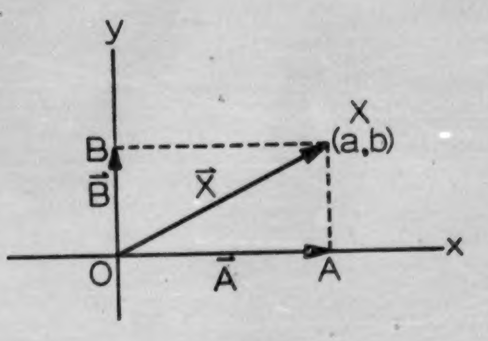


Figure 3-23

As before, we have the component vectors $a\mathbf{i} = \overrightarrow{A}$, and $b\mathbf{j} = \overrightarrow{B}$.

We now wish to extend this concept of component vectors. Consider any non-zero origin-vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$ to points X and Y respectively. Let the perpendicular from X to $\overleftrightarrow{OY}$ meet $\overleftrightarrow{OY}$ in point P as indicated in Figure 3-24. Then the vectors $\overrightarrow{m}$ and $\overrightarrow{n}$ corresponding to $\overrightarrow{OP}$ and $\overrightarrow{PX}$ are called the component vectors of $\overrightarrow{X}$ with respect to $\overrightarrow{Y}$. This idea is not restricted to origin-vectors.

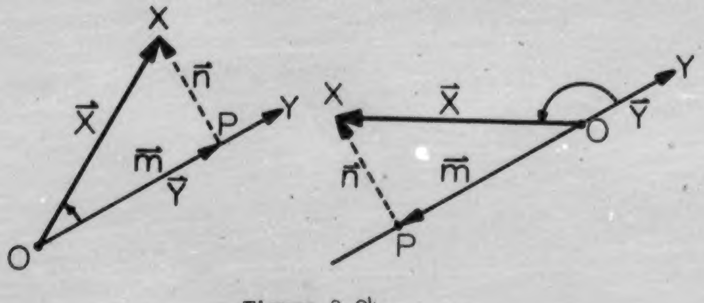


Figure 3-24

This extension of the concept of components of vectors is often helpful in physical and geometric applications, where these ideas are discussed in terms of the resolution of a vector into vector components. In the above discussion, we say that we resolve $\overrightarrow{X}$ into vector components m and n respectively parallel and perpendicular to $\overrightarrow{Y}$.

From the definition of the inner product of two vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$, we have

(1) the component of $\overrightarrow{X}$ in the direction of $\overrightarrow{Y}$,

$$\overrightarrow{X} \cos \theta = \frac{\overrightarrow{X} \cdot \overrightarrow{Y}}{|\overrightarrow{Y}|} = \overrightarrow{X} \cdot \frac{\overrightarrow{Y}}{|\overrightarrow{Y}|} \quad \text{where}$$

$\frac{\overrightarrow{Y}}{|\overrightarrow{Y}|}$ represents the unit vector along the $\overrightarrow{Y}$ direction.

(2) the component of $\overrightarrow{Y}$ in the direction of $\overrightarrow{X}$,

$$\overrightarrow{Y} \cos \theta = \frac{\overrightarrow{X} \cdot \overrightarrow{Y}}{|\overrightarrow{X}|} = \overrightarrow{Y} \cdot \frac{\overrightarrow{X}}{|\overrightarrow{X}|} \quad \text{where}$$

$\frac{\overrightarrow{X}}{|\overrightarrow{X}|}$ represents the unit vector along the $\overrightarrow{X}$ direction.

Exercises 3-8 and 3-9

1. Verify Theorem 3-14 (b) for the vectors

$$\overrightarrow{X} = [2,4] \ , \ \overrightarrow{Y} = [-1,-3] \ \text{ and } \ t = 5 \ .$$

2. If $\overrightarrow{X} = [x_1,x_2]$ and $\overrightarrow{Y} = [y_1,y_2]$, prove that $(t\overrightarrow{X}) \cdot \overrightarrow{Y} = \overrightarrow{X} \cdot (t\overrightarrow{Y})$ for any scalar t .

3. Prove the corollary of Theorem 3-14.

4. (a) Supply the reasons for each step of the proof of the theorm in Example 1 following Theorem 3-14.
   (b) Same as (a) for the theorem in Example 2.

5. Find the area of the triangle determined by $\overrightarrow{A} = [3,-1]$ and $\overrightarrow{B} = [2,6]$ and check your result by any method.

6. Given $\overrightarrow{A} = 2i - 3j$ and $\overrightarrow{B} = -2i + j$. Find the component of
   (a) $\overrightarrow{A}$ upon $\overrightarrow{B}$
   (b) $\overrightarrow{B}$ upon $\overrightarrow{A}$

7. Given a vector representing a wind of 30 mph. from the southwest. Locate this vector in a coordinate plane where the positive side of the y-axis is considered to lie in the north direction. Resolve this vector into its $\underline{m}$ and $\underline{n}$ components (as described in Figure 3-23) with respect to:

   (a) the x and y axes.

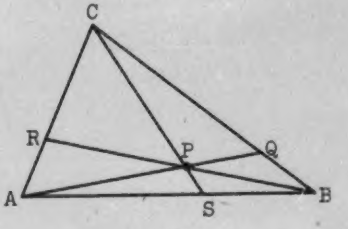
   (b) the line $\theta = 15^\circ$.

   (c) the vector $\overrightarrow{A} = [10,15]$.

Challenge Problems

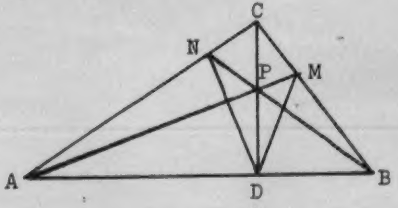
1. (Ceva's Theorem) Let P be any point not on triangle ABC . Let $\overrightarrow{AP}$, $\overrightarrow{BP}$, $\overrightarrow{CP}$ intersect $\overline{BC}$, $\overline{AC}$, $\overline{AB}$ respectively at Q, R, S . Show that

$$\frac{d(A,S)}{d(S,B)} \cdot \frac{d(B,Q)}{d(Q,C)} \cdot \frac{d(C,R)}{d(R,A)} = 1$$



2. In triangle ABC , let $\overline{CD} \perp \overline{AB}$ and let P be any point on $\overline{CD}$. Let $\overrightarrow{AP}$ intersect $\overline{BC}$ at M and $\overrightarrow{BP}$ intersect $\overline{AC}$ at N . Show that $\angle CDN = \angle CDM$.

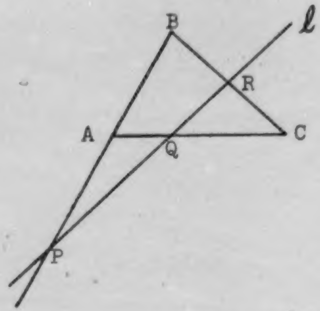
   (Hint. Take D to be 0 .)



3. (Menelaus' Theorem) Let ℓ be any line which does not pass through any vertex of triangle ABC . Let ℓ intersect $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$ respectively at P, Q, R .

   Show that

$$\frac{d(A,Q)}{d(Q,C)} \cdot \frac{d(C,R)}{d(R,B)} \cdot \frac{d(B,P)}{d(P,A)} = 1 .$$



4. (a) Prove algebraically

$$(x_1y_1 + x_2y_2)^2 \leq (x_1{}^2 + x_2{}^2)(y_1{}^2 + y_2{}^2) .$$

NOTE: This is a case of Schwarz's inequality, another form of which is

$$(x_1y_1 + x_2y_2 + x_3y_3)^2 \leq (x_1{}^2 + y_2{}^2 + x_3{}^2)(y_1{}^2 + y_2{}^2 + y_3{}^2) .$$

(b) Write these in vector notation.

(c) What geometric interpretation can be made for the case in which the left and right members are equal.

3-10. Summary and Review Exercises.

The chapter just concluded dealt with vectors and their applications. After reviewing some basic ideas about directed line segments (objects with both direction and magnitude), a vector was defined as an infinite set of equivalent directed line segments. The Origin-Principle allowed us to relate a vector to any point in space as an origin. We found it useful to select the origin-vector, that member of each set with its initial point at the origin, as the simplest representative of a vector. The unit vector and zero vector were defined and the term scalar introduced.

The next step in setting up an algebra of vectors was taken when the equality of vectors was defined in accordance with common practice. The operations of addition and subtraction of vectors and the product of a vector by a scalar were defined. The last concept made it possible to state that two vectors are parallel if and only if one is a scalar multiple of the other. The Origin-Principle related operations with vectors to the corresponding operations with their respective origin-vectors.

It was then proved that the commutative and associative laws hold for the addition of vectors. Scalar multiplication satisfied the associative law $(rs)\vec{P} = r(s\vec{P})$ and the distributive laws $r(\vec{P} + \vec{Q}) = r\vec{P} + r\vec{Q}$ and $(r + s)\vec{P} = r\vec{P} + s\vec{P}$. The zero vector $\vec{0}$ has the usual properties of the additive identity; the additive inverse, $-\vec{P}$, is defined by $\vec{P} + (-\vec{P}) = \vec{0}$.

The definition of a linear combination of vectors made it possible to prove some basic theorems about vectors. Theorem 3-5 stated that in a plane any vector can be expressed in terms of any two non-parallel and non-zero vectors. After the study of vector components, it was pointed out that any vector can be represented as a linear combination of the unit vectors

$i = [1,0]$ and $j = [0,1]$. Theorem 3-7 made it possible to determine if a point P lies on the line passing through the terminal points of two distinct vectors $\vec{A}$ and $\vec{B}$ which do not lie on the same line by proving that $\vec{P} = (1 - r)\vec{A} + r\vec{B}$. Sets of points on a given line could now be given a vector characterization. Theorem 3-8 offered a second method for dividing a line segment in a given ratio.

Vector components play a basic role in the application of vectors. The operations on vectors were defined in terms of these components. If $\vec{X} = [a,b]$, $\vec{Y} = [c,d]$, then $\vec{X} + \vec{Y} = [a + c\ ,\ b + d]$ and $r\vec{X} = [ra,rb]$.

The inner product of two vectors was defined by $\vec{X} \cdot \vec{Y} = |\vec{X}|\,|\vec{Y}| \cos \theta$ where $\angle\theta$ is the angle between the two vectors, with $0 \leq \theta \leq \pi$. It was then proved that if $\vec{X} = [x_1,x_2]$ and $\vec{Y} = [y_1,y_2]$, then $\vec{X} \cdot \vec{Y} = x_1y_1 + x_2y_2$. A physical application was presented in the concept of work in physics. An important theorem is that two vectors, $\vec{X}$ and $\vec{Y}$, are perpendicular if and only if $\vec{X} \cdot \vec{Y} = 0$. The inner product has the following properties:

(1) $\vec{X} \cdot (\vec{Y} + \vec{Z}) = \vec{X} \cdot \vec{Y} + \vec{X} \cdot \vec{Z}$.

(2) $(t\vec{X}) \cdot \vec{Y} = \vec{X} \cdot (t\vec{Y}) = t(\vec{X} \cdot \vec{Y})$ where t is a scalar.

(3) $\vec{X} \cdot (a\vec{Y} + b\vec{Z}) = a(\vec{X} \cdot \vec{Y}) + b(\vec{X} \cdot \vec{Y})$ where a and b are scalars.

The inner product has many applications in geometry. We showed how it could be used to determine an angle between vectors, to find the area of the triangle determined by two vectors with a common initial point, to prove that the diagonals of a rhombus are perpendicular, and to show that the altitudes of a triangle are concurrent. The chapter concluded with a discussion of the resolution of vectors. This concept has considerable application in physical problems.

In the following chapter which deals with methods of proof in analytic geometry, there will be more proofs applying vector methods to geometric problems. In Chapter 8 there will be a brief introduction to vectors in a three dimensional space.

Review Exercises

1. If $\vec{A} = [3,-5]$, $\vec{B} = [-1,6]$, $\vec{C} = [2,3]$, find $\vec{X}$ in component form such that

   (a) $\vec{A} + \vec{B} = \vec{C} + \vec{X}$

   (b) $2\vec{A} + 3\vec{B} = 4\vec{C} + 5\vec{X}$

   (c) $2(\vec{A} - \vec{B}) = 3(\vec{C} - \vec{X})$

   (d) $\vec{A} + 2\vec{X} = \vec{B} + \vec{C} - \vec{X}$

   (e) $3(\vec{X} + \vec{B}) = 2(\vec{X} - \vec{C})$

   (f) $\vec{X} + 2(\vec{X} + \vec{A}) + 3(\vec{X} + \vec{B}) = 0$

2. Prove Theorem 3-3.

3. Prove Theorem 3-4.

4. Let $\vec{A} = [2,3]$, $\vec{B} = [3,-2]$, $\vec{C} = [-1,3]$. Find in component form, the single vector equal to

   (a) $2\vec{A} + 3\vec{B} - \vec{C}$

   (b) $\vec{A} - 2\vec{B} + 3\vec{C}$

   (c) $2(\vec{A} + \vec{B}) - 3(\vec{B} - \vec{C})$

   (d) $5(\vec{A} - \vec{C}) + 3(\vec{C} - \vec{A})$

   (e) $3(\vec{A} + \vec{B} - \vec{C}) + 2(\vec{A} - \vec{B} + \vec{C})$

   (f) $5(\vec{C} - \vec{A} + \vec{B}) - 3(\vec{B} + \vec{A} - \vec{C})$

5. Use the values of $\vec{A}$, $\vec{B}$, $\vec{C}$, as in Exercise 4, and find $\vec{X}$ in component form so that

   (a) $\vec{A} + \vec{B} = \vec{C} + \vec{X}$

   (b) $2\vec{A} + 3\vec{B} = 4\vec{C} + 5\vec{X}$

   (c) $2(\vec{A} - \vec{B}) = 3(\vec{C} - \vec{X})$

   (d) $\vec{A} + 2\vec{X} = \vec{B} + \vec{C} - \vec{X}$

   (e) $3(\vec{X} + \vec{B}) = 2(\vec{X} - \vec{C})$

   (f) $\vec{X} + 2(\vec{X} + \vec{A}) + 3(\vec{X} + \vec{B}) = 0$

6. Use the values of $\vec{A}$, $\vec{B}$, $\vec{C}$, as in Exercise 4, and find the numerical value of

   (a) $\vec{A} \cdot \vec{B}$

   (b) $2\vec{A} \cdot 3\vec{B}$

   (c) $3\vec{A} \cdot (\vec{B} + \vec{C})$

   (d) $2\vec{B} \cdot (3\vec{A} + 2\vec{C})$

   (e) $(\vec{A} + \vec{B}) \cdot (\vec{A} - \vec{B})$

   (f) $(2\vec{B} + 3\vec{C}) \cdot (2\vec{B} - 3\vec{C})$

   (g) $(3\vec{A} + 5\vec{B}) \cdot (3\vec{B} - 2\vec{C})$

   (h) $(\vec{A} + \vec{B} - \vec{C}) \cdot (\vec{B} - \vec{A} + \vec{C})$

   (i) $(2\vec{A} - 3\vec{B} + 4\vec{C}) \cdot (5\vec{A} - 2\vec{C} + 4\vec{B})$

   (j) $\vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} + \vec{C} \cdot \vec{C}$

7. Use the values of A , B , C , as in Exercise 4, and find the numerical values of

   (a) $|\vec{A}| + |\vec{B}|$

   (b) $|2\vec{A}| + |3\vec{C}|$

   (c) $2|\vec{A}| + 3|\vec{C}|$

   (d) $|3\vec{B}| - |4\vec{A}|$

   (e) $|\vec{A} - \vec{B}|$

   (f) $|2\vec{A} + 3\vec{C}|$

   (g) $|3\vec{B} - 4\vec{A}|$

   (h) $|\vec{A}|^2 - |\vec{B}|^2$

   (i) $|\vec{A}|^2 + |\vec{B}|^2 + |\vec{C}|^2$

   (j) $|2\vec{A}|^2 + |3\vec{B}|^2 + |4\vec{C}|^2$

   (k) $|2\vec{A} + 3\vec{B} + 4\vec{C}|^2$

   (l) $|\vec{A} - \vec{B}|^2$

   (m) $2|\vec{A}|^2 + 3|\vec{B}|^2 + 4|\vec{C}|^2$

   (n) $|\vec{A}|^2 + 2|\vec{A}||\vec{B}| + |\vec{B}|^2$

8. If $i = [1,0]$ and $j = [0,1]$, we may express the vectors of Exercise 4 thus: $\vec{A} = 2i + 3j$, $\vec{B} = 3i - 2j$, $\vec{C} = -i + 3j$. In each part of Exercise 4, restate the original problem in terms of i and j; then, carry out your computations and express your results in terms of these components.

9. (Refer to Exercises 8 and 4 above.) Restate, in each part of Exercise 5, the problem and the solution in terms of i and j components.

10. (Refer to Exercises 8 and 4 above.) Restate, in each part of Exercise 6, the problem and the solution in terms of i and j components.

11. Given $A = (4,1)$, $B = (2,5)$, $C = (-2,3)$, and $D = (0,-4)$.

    (a) Find the angle measure of $\angle ABC$, $\angle BCD$, $\angle CDA$, and $\angle DAB$; check your results.

    (b) Using O as the origin, find the areas of $\triangle OAB$, $\triangle OBC$, and $\triangle OAC$.

    (c) Use the results from part (b) to find the area of $\triangle ABC$.

12. Try to develop, with the methods of this chapter, a formula for the area of $\triangle ABC$, where $A = (a_1,a_2)$, $B = (b_1,b_2)$, $C = (c_1,c_2)$.

13. Find the area of the parallelogram in which $\overline{OA}$ and $\overline{OB}$ are adjacent sides. Can you apply these results to an earlier exercise in this set?

14. Find the vector representation of an exterior point of division which divides the directed segment (R,S) in the ratio $\frac{a}{b}$ where:

    (a) $\vec{R} = [2,-1]$, $\vec{S} = [-1,3]$, and $\frac{a}{b} = -2$

    (b) $\vec{R} = [-1]$, $\vec{S} = [2]$, and $\frac{a}{b} = -\frac{1}{2}$

    (c) $\vec{R} = [2,3,1]$, $\vec{S} = [1,-2,4]$, and $\frac{a}{b} = -3$

    (d) $\vec{R} = [-9,7]$, $\vec{S} = [3,-2]$, and $\frac{a}{b} = -\frac{1}{3}$

*15. Given the triangle ABC with $\vec{A} = [2,3]$, $\vec{B} = [-1,2]$, and $\vec{C} = [1,4]$.

    (a) Describe the triangular region, its interior, and the triangle itself, using these vectors and two scalars.

    (b) Show that $[1,3]$ is a vector whose terminal point is an interior point of the triangle.

    (c) Show that $[1,1]$ is a vector whose terminal point is an exterior point of the triangle.

    *(d) Show that the segment joining the points described in (b) and (c) intersects the triangle.

*16. Consider the convex quadrilateral ABCD with $\vec{A} = [2,3]$, $\vec{B} = [-1,2]$, $\vec{C} = [1,4]$, and $\vec{D} = [2,4]$. Find an expression for the polygonal region ABCD using these vectors and three scalars.

*17. Given the four vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$, whose terminal points are not coplanar, find an expression for the tetrahedral region ABCD in terms of these vectors and three scalars.

18. Find the measure of the angles formed by the intersection of the lines

    (a) $2x + 3y - 8 = 0$ and $3x - 2y + 4 = 0$.
    (b) $5x + y - 2 = 0$ and $2x - y + 6 = 0$.
    (c) $x + y + 3 = 0$ and
    (d) $x + 2y = 0$ and $x = 4$

19. Points $A = (1,0)$, $B = (5,-2)$, and $C = (3,4)$ are the vertices of a triangle. Find the measure of each angle of $\triangle ABC$.

20. Given points $P = (-3,-8)$, $Q = (14,9)$, $R = (4,9)$, and $S = (-3,2)$. Find the measure of each angle of quadrilateral PQRS , and name the figure.

Chapter 4

PROOFS BY ANALYTIC METHODS

4-1. Introduction.

One of the satisfactions we hope you will gain from your study of analytic geometry is the realization that you have some very powerful tools for solving many seemingly difficult or impossible problems. We can demonstrate this, even so early in our work, by observing the simplicity and directness of analytic proofs for some theorems from plane geometry and trigonometry. You will recall many of these theorems, and you also may recall some of the struggles which resulted from using synthetic methods on these problems.

By increasing the number of methods available to solve problems, we create another problem--the uncertainty as to which method to use in a given situation. We shall sometimes ask you to use a particular method so that you may develop competence and confidence in its use. A tennis player may, in order to strengthen his backhand, be encouraged to use it temporarily more than he would in normal play. Your uncertainty and discomfort with a new method will last only until you have mastered it. You should understand also that even a competent mathematician may start with one method and discover later that it is not as convenient as another method. As you study the examples in this chapter, you should watch for clues to the reasons for choosing one method rather than another. Careful observation at this point will smooth the way as you proceed.

For the purposes of this chapter we assume that you know the kinds and basic properties of common geometric figures and that diagonals, medians, and the like, have been defined. These items, as well as the theorems to be discussed, may be reviewed in SMSG Geometry, Intermediate Mathematics, or some equivalent source.

4-2. Proofs Using Rectangular Coordinates.

Let us now prove some geometric theorems in rectangular coordinates.

Example 1. Prove: The median to the base of an isosceles triangle is perpendicular to the base. We might find the triangle placed in relation to the coordinate axes, as in Figure 4-1, with $\overline{AC} \cong \overline{BC}$ and with D the midpoint of $\overline{AB}$. From an analytic point of view, to prove $\overline{CD} \perp \overline{AB}$ we must show that the product of the slope of $\overline{AB}$ and the slope of $\overline{CD}$ is -1 .

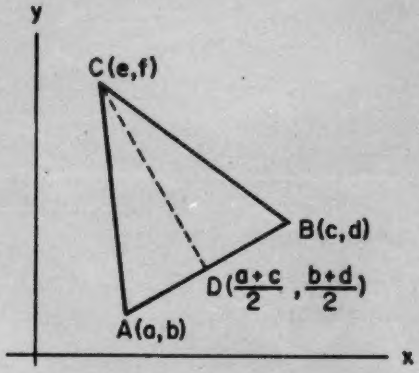


Figure 4-1

In order to ensure that the triangle is a general one we might select coordinates as follows: $A = (a,b)$, $B = (c,d)$, $C = (e,f)$. It follows that midpoint $D = (\frac{a+c}{2}, \frac{b+d}{2})$. By hypothesis $d(A,C) = d(B,C)$. We apply the distance formula to obtain

$$\sqrt{(a-e)^2 + (b-f)^2} = \sqrt{(c-e)^2 + (d-f)^2} ,$$

$$a^2 - 2ae + e^2 + b^2 - 2bf + f^2 = c^2 - 2ce + e^2 + d^2 - 2df + f^2 , \text{ or}$$

$$(1) \qquad a^2 - 2ae + b^2 - 2bf = c^2 - 2ce + d^2 - 2df .$$

We next calculate slopes. The slope of $\overline{CD}$ is $\dfrac{\frac{b+d-2f}{2}}{\frac{a+c-2e}{2}}$ and the slope of $\overline{AB}$ is $\dfrac{b-d}{a-c}$.

The product of the two slopes is

$$\frac{b^2 + bd - 2bf - bd - d^2 + 2df}{a^2 + ac - 2ae - ac - c^2 + 2ce} = \frac{b^2 - 2bf - d^2 + 2df}{a^2 - 2ae - c^2 + 2ce} .$$

Equation (1) can be written as

$$(2) \qquad a^2 - 2ae - c^2 + 2ce = -b^2 + 2bf + d^2 - 2df .$$

Substituting the right member of (2) into the denominator of the product of the slopes, we obtain

$$\frac{b^2 - 2bf - d^2 + 2df}{-b^2 + 2bf + d^2 - 2df} = -1 ;$$

hence, the theorem is proved.

It would be discouraging indeed if all of our coordinate proofs involved as much algebraic manipulation as exhibited in this example. Fortunately, this is not the case, and you may already see what can be done to simplify the algebra. It was not necessary to choose the coordinates as we did.

The properties of geometric figures depend upon the relations of the parts and not upon the position of the figure as a whole. Therefore, in our example, since only the triangle and not its location is specified, we could just as well select a coordinate system in which A is the origin and B lies on the positive side of the x-axis. This situation is illustrated in Figure 4-2. We now may have the following coordinates for the points: $A = (0,0)$, $B = (a,0)$, $C = (b,c)$, $D = (\frac{a}{2},0)$. *Note that several of the coordinates are zero.* This is the feature which simplifies the algebra in our theorems, and this desirable goal provides us with a general guide in choosing coordinate axes for all our problems.

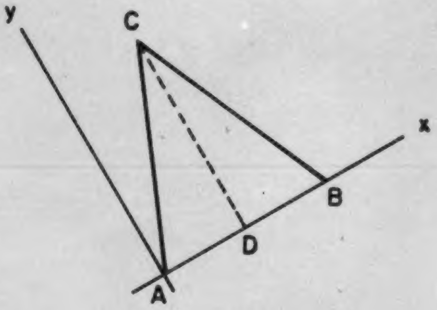


Figure 4-2

In actual practice we are more likely to make a drawing with the axes oriented as in Figure 4-3. This leads us to consider two methods of relating a geometric figure to a set of axes.

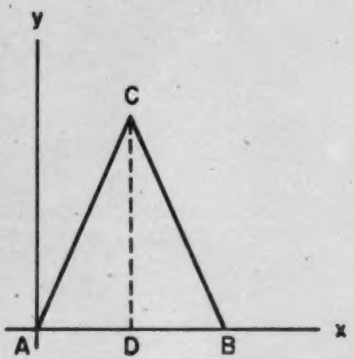


Figure 4-3

The method we have just described, that of assigning coordinates to a given geometric figure, is based upon the properties of coordinate systems developed in Chapter 2. Another method in common use employs the principles of rigid motion in which geometric objects are "moved" to more suitable locations without changing their size or shape. With respect to our current example, we would arrive at Figure 4-3 through this second method by assuming a fixed coordinate system upon which we place $\triangle ABC$ so that A coincides with the origin and B is placed on the positive side of the x-axis. The difference in the methods is largely one of viewpoint.

Another device which you will find useful can be illustrated by assigning coordinates to the vertices of $\triangle ABC$ in Figure 4-3 as follows: $A = (0,0)$, $B = (2a,0)$, $C = (b,c)$. The reason for using $2a$ for the abscissa of B is that we now have $D = (a,0)$; and we can complete the algebra without so much calculation involving fractions. The principle here is that a few minutes of foresight may save hours of patience.

Sometimes we pay a small price for the simplicity we gain. For example, the choice of coordinates suggested in the previous paragraph leads to trouble regarding the slopes. Although the slope of $\overline{AB}$ can be found to be zero, $\overline{CD}$ does not have a slope, since $a = b$. (Use the distance formula with $d(A,C) = d(B,C)$ to verify this.) Nevertheless, the problem has been simplified, for this means that $\overline{AB}$ is horizontal and $\overline{CD}$ is vertical, and this is also a condition for perpendicularity.

You might have chosen a coordinate system in which $\overline{AB}$ is on the x-axis but D is the origin. This is a fine choice. As you can see in Figure 4-4, if we choose $A = (a,0)$, then $B = (-a,0)$. It remains for us to prove that C lies on the y-axis. Let $C = (b,c)$ and use the distance formula in $d(A,C) = d(B,C)$. You can show that $b = 0$; hence, C lies on the y-axis and $\overline{CD} \perp \overline{AB}$.

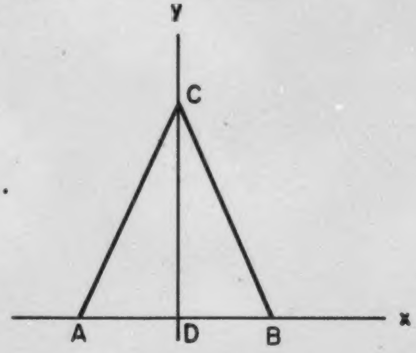


Figure 4-4

Let us summarize the procedures we have seen in this example. Usually there are more ways than one to attack any given problem, but certain general steps can be outlined. It was natural and useful in this example to use rectangular coordinates, since we were concerned with midpoints, lengths, and perpendicularity. Other situations we meet later may lead naturally to vectors or polar coordinates. In the cases for which we decide to use rectangular coordinates, we might follow the outline suggested below.

(a) Choose a coordinate system (or place the figure on one) so as to simplify the algebraic processes. Often this means having a vertex of the figure at the origin and one of its sides on the x-axis.

(b) Assign coordinates to points of the figure so as to accommodate the hypothesis as simply and clearly as possible. That is, make the figure sufficiently, but not unnecessarily, general.

(c) If possible, state the hypothesis and conclusion in a way that will correspond closely to the algebraic procedures being used.

(d) Plan an algebraic proof. Watch for opportunities to employ the distance, midpoint, and slope formulas.

Let us try another theorem from plane geometry.

Example 2. Prove: The diagonals of a parallelogram bisect each other.

Following the outline of our procedures, (a) to (c), we represent a parallelogram in a drawing and orient it with respect to the axes as in Figure 4-5. We let $A = (0,0)$ and $B = (a,0)$. The question of choosing coordinates for C and D can stand some discussion. The coordinates of C and D are not independent of those of A and B, nor are they independent of each other. How much can we assume about a parallelogram? We know by definition that the opposite sides of a parallelogram are parallel. This enables us to see at once that C and D have the same ordinate. Furthermore, since $\overline{BC} \parallel \overline{AD}$, their slopes are equal. This suggests that we use the slope formula to obtain a relation between the abscissas of C and D; namely, that the abscissa of C is the abscissa of B plus the abscissa of D. Thus we write $D = (b,c)$ and $C = (a + b,c)$. If we are allowed to use

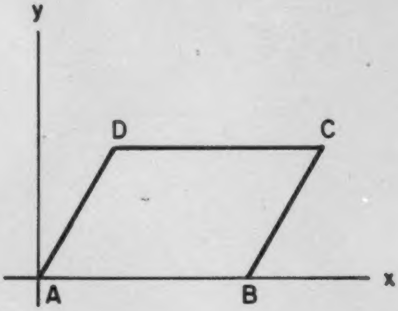


Figure 4-5

the property of a parallelogram that the opposite sides have equal lengths, then we shall reach the same conclusion more readily.

Some people prefer to employ these elementary properties of the common figures; others choose to assume no more than the definitions. For the purposes of this section we shall agree that we may use the properties ascribed to geometric figures by their definitions and by the theorems listed in Exercises 4-2, taking these theorems in the order in which they are listed. Our current example would be listed after Exercise 4 so the conclusion of Exercise 4 would be available to us when we chose coordinates for Figure 4-5.

The conclusion of our example is reached quickly. We are required to prove that the diagonals bisect each other. This means that each diagonal intersects the other at its midpoint. An application of the midpoint formula shows that the midpoint of each diagonal is $(\frac{a+b}{2}, \frac{c}{2})$.

We conclude this section with a challenge. Try to prove the following theorem by synthetic methods, and compare your proof with the one suggested below.

Example 3. Prove: If two medians of a triangle are congruent, the triangle is isosceles.

We prefer to use coordinates. The triangle must not be assumed to be isosceles, so we assign coordinates in Figure 4-6 as follows: $A = (2a,0)$, $B = (2b,0)$, $C = (0,2c)$. Let $M = (a,c)$ be the midpoint of $\overline{AC}$, and let $N = (b,c)$ be the midpoint of $\overline{BC}$. Next we shall express the hypothesis, $d(A,N) = d(B,M)$, in terms of the distance formula. You are encouraged to state the desired conclusion and to complete the details of the proof.

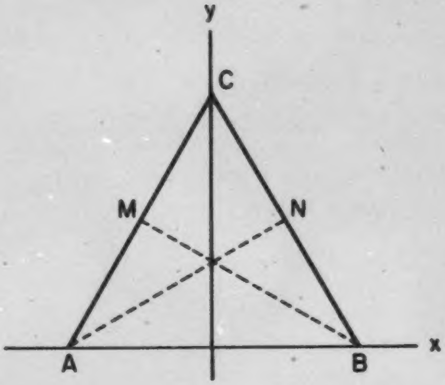


Figure 4-6

Exercises 4-2

The following exercises are theorems selected from the usual development of plane geometry. You are to prove these theorems in rectangular coordinates, using the "ground rules" we have outlined.

1. The line segment joining the midpoints of two sides of a triangle is parallel to the third side and has length equal to one-half the length of the third side.
2. If a line bisects one side of a triangle and is parallel to a second side, it bisects the third side.
3. The locus of points equidistant from two points is the perpendicular bisector of the line segment joining the two given points.
4. The opposite sides of a parallelogram have equal length.
5. If two sides of a quadrilateral have equal length and are parallel, the quadrilateral is a parallelogram.
6. If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.
7. If the diagonals of a parallelogram have equal length, the parallelogram is a rectangle.
8. The diagonals of a rhombus are perpendicular.
9. If the diagonals of a parallelogram are perpendicular, the parallelogram is a rhombus.
10. The line segments joining in order the midpoints of the successive sides of a quadrilateral form a parallelogram.
11. The line segments joining the midpoints of the opposite sides of a quadrilateral bisect each other.
12. The diagonals of an isosceles trapezoid have equal length.
13. The median of a trapezoid is parallel to the bases and has length equal to one-half the sum of the lengths of the bases.
14. If a line bisects one of the nonparallel sides of a trapezoid and is parallel to the bases, it bisects the other nonparallel side.

15. In any triangle, the square of the length of a side opposite an acute angle is equal to the sum of the squares of the lengths of the other two sides minus twice the product of the length of one of the two sides and the length of the projection of the other on it.

16. The medians of a triangle are concurrent in a point that divides each of the medians in the ratio 2:1 .

17. The altitudes of a triangle are concurrent.

*18. A line through a fixed point P intersects a fixed circle in points A and B . Find the locus of the midpoint of $\overline{AB}$. (Consider three possible positions for P relative to the fixed circle.)

4-3. Proofs Using Vectors.

We shall now prove several theorems of geometry by vector methods. Some of the proofs are more difficult than those using methods discussed in your geometry course or in the preceding section. Others are accomplished more simply or concisely. In any case, the experience will be of great help in future mathematics courses and in applications to science or engineering. It will contribute toward your general ability to solve problems by giving you an additional tool and approach.

We shall demonstrate these approaches by solving several problems in detail.

Example 1. Prove that the median of a trapezoid is parallel to the bases and has length equal to one-half the sum of the lengths of the bases.

We first draw and label a trapezoid ABCD with $\overline{AB} \parallel \overline{CD}$ and with E and F the respective midpoints of $\overline{AD}$ and $\overline{BC}$. If we were using a rectangular coordinate system in this proof, we probably would choose the axes as in Figure 4-7. But since we are using a vector proof, we do not need the axes at all. In fact, because the origin vectors would not give us any advantage in the proof, neither do we specify an origin.

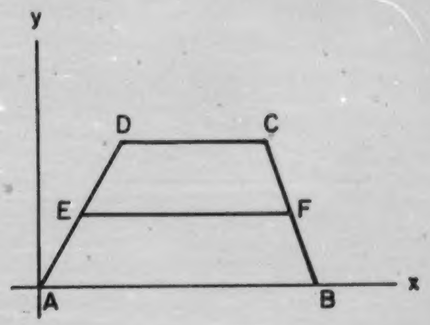


Figure 4-7

A vector drawing for the problem might then appear as in Figure 4-8.

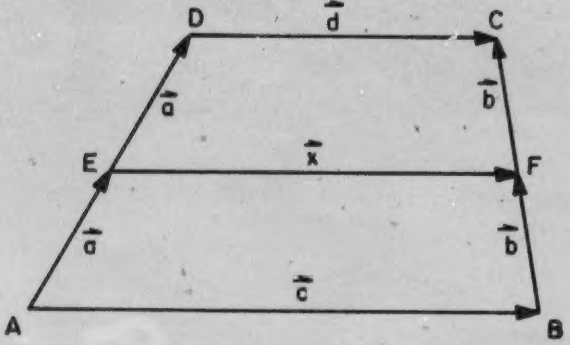


Figure 4-8

Something should be said about our choice of vector representation. Since E is the midpoint of $\overline{AD}$, if we represent $\overline{AE}$ by $\vec{a}$, then $\overline{ED}$ may also be represented by $\vec{a}$. Similarly, we choose $\vec{b}$ on the other non-parallel side. $\vec{c}$ and $\vec{d}$ represent the bases, and $\vec{x}$ represents median $\overline{EF}$. We are to prove

$$d(E,F) = \frac{1}{2}\Big(d(A,B) + d(C,D)\Big) \quad \text{and} \quad \vec{x} \,||\, \vec{c} \quad \text{and} \quad \vec{x} \,||\, \vec{d}\,.$$

Since one may "move" from E to F by going directly there, or by going through D and C, or by going through A and B, we have

$$\vec{x} = \vec{a} + \vec{d} - \vec{b}$$

and

$$\vec{x} = -\vec{a} + \vec{c} + \vec{b}\,;$$

therefore,

$$2\vec{x} = \vec{c} + \vec{d}\,.$$

Note again that when "moving" around a vector diagram, we add vectors which have the same sense of direction as our motion, and we subtract vectors which have the opposite sense of direction of our motion.

By the definition of parallel vectors, if $2\vec{x} = \vec{c} + \vec{d}$, then $\vec{x} \,||\, (\vec{c} + \vec{d})$; since it is given that $\vec{c} \,||\, \vec{d}$, it follows that $\vec{x} \,||\, \vec{c}$ and $\vec{x} \,||\, \vec{d}$. Furthermore, if $2\vec{x} = \vec{c} + \vec{d}$, then

$$|\vec{x}| = \frac{1}{2}\Big(|\vec{c}| + |\vec{d}|\Big) \quad \text{or} \quad d(E,F) = \frac{1}{2}\Big(d(A,B) + d(C,D)\Big)\;;$$

hence, the theorem is proved. You may wish to investigate what happens to the proof if you alter the direction of any of the vectors in the diagram.

__Example 2__. Show that the midpoints of the sides of a quadrilateral are the vertices of a parallelogram.

This situation is depicted by Figure 4-9 in which P, Q, R, and S are the given midpoints of the sides of quadrilateral ABCD. Once we choose an origin, each point of the figure determines an origin-vector. (It might be profitable for you to copy the figure on a piece of paper, select some point as an origin, and draw the origin-vectors to the vertices.)

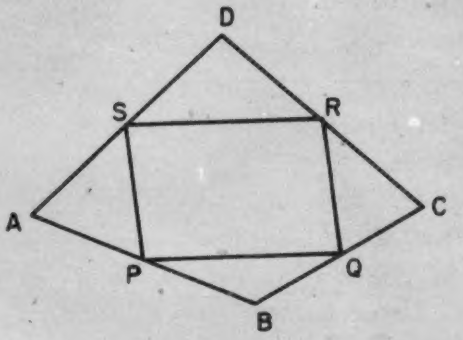


Figure 4-9

A portion of the figure with a set of origin-vectors is shown in Figure 4-10. We have also identified the vectors from A to P and from P to B in order to make use of the fact that $d(A,P) = d(P,B)$.

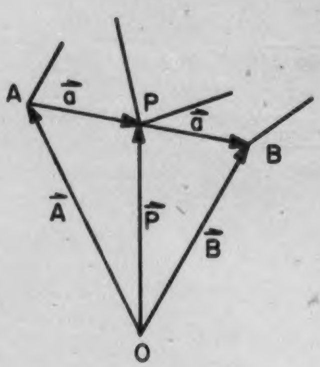


Figure 4-10

Since $\overrightharpoon{P} = \overrightharpoon{A} + \overrightharpoon{a}$

and $\overrightharpoon{P} = \overrightharpoon{B} - \overrightharpoon{a}$,

$2\overrightharpoon{P} = \overrightharpoon{A} + \overrightharpoon{B}$

or $\overrightharpoon{P} = \frac{1}{2}(\overrightharpoon{A} + \overrightharpoon{B})$.

Similarly, $\overrightharpoon{Q} = \frac{1}{2}(\overrightharpoon{B} + \overrightharpoon{C})$,

$\overrightharpoon{R} = \frac{1}{2}(\overrightharpoon{C} + \overrightharpoon{D})$,

$\overrightharpoon{S} = \frac{1}{2}(\overrightharpoon{A} + \overrightharpoon{D})$.

(Had we not been interested in calling your attention to an application of vector addition, we would have obtained the same results from the Point of Division Theorem.)

We next note that vector $\vec{P} - \vec{Q}$ is equal to vector $\vec{S} - \vec{R}$ because both are equal to $\frac{1}{2}(\vec{A} - \vec{C})$. But why did we choose an expression like $\vec{P} - \vec{Q}$? There is a good reason for the choice. The line on vector $\vec{P} - \vec{Q}$ is parallel to $\overleftrightarrow{PQ}$, and remember that we are to show that certain segments are parallel.

In order to see the importance of $\vec{P} - \vec{Q} = \vec{S} - \vec{R}$, let us take a closer look at this situation, using a different origin. Suppose we isolate the lower part of Figure 4-9 containing points P, B, and Q as in Figure 4-11. If we choose B as the origin and E so that B is the midpoint of $\overline{QE}$, then we have vectors as marked on the diagram. The vector from Q to P is $-\vec{q} + \vec{p}$ which equals $\vec{P} - \vec{Q}$ and is therefore equal to $\vec{T}$. It follows then that the line on vector $\vec{P} - \vec{Q}$ is parallel to $\overleftrightarrow{PQ}$. Similarly the line on vector $\vec{S} - \vec{R}$ is parallel to $\overleftrightarrow{SR}$; and, since $\vec{P} - \vec{Q}$ is equal to and, consequently, parallel to $\vec{S} - \vec{R}$, we conclude that $\overline{PQ} \parallel \overline{SR}$. In the same way we show that $\overline{PS} \parallel \overline{QR}$, and PQRS is a parallelogram.

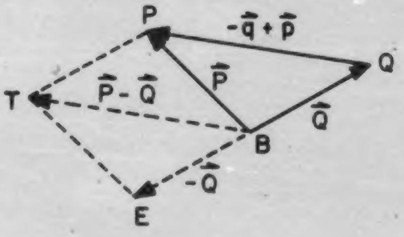


Figure 4-11

Example 3. Prove that the medians of a triangle intersect in a point which is a point of trisection of each median.

Solution. Let ABC be the triangle and P, Q, and R the midpoints of its sides as shown in Figure 4-12.

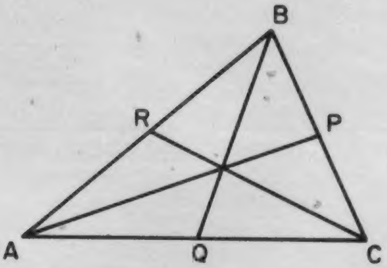


Figure 4-12

By the Origin Principle we may place the origin wherever we wish. If we are successful in proving the medians concurrent, the point of intersection would be an ideal choice for the origin, for then each origin-vector to a vertex would be collinear with the origin-vector to the midpoint of the opposite side.

We cannot assume all three medians concurrent, but we can let the origin O be the intersection of $\overline{AP}$ and $\overline{BQ}$. Then to prove that $\overline{CR}$ contains this point, we must prove that $\overrightarrow{R}$ and $\overrightarrow{C}$ are collinear, or that $\overrightarrow{R}$ is a scalar multiple of $\overrightarrow{C}$.

Proof. Let the origin be the intersection of $\overline{AP}$ and $\overline{BQ}$. Since P and Q are midpoints, and since $\overrightarrow{P}$ and $\overrightarrow{Q}$ are collinear with $\overrightarrow{A}$ and $\overrightarrow{B}$ respectively, we may write

(1) $\overrightarrow{P} = \frac{1}{2}(\overrightarrow{B} + \overrightarrow{C}) = x\overrightarrow{A}$

(2) $\overrightarrow{Q} = \frac{1}{2}(\overrightarrow{A} + \overrightarrow{C}) = y\overrightarrow{B}$.

If we subtract Equation (2) from Equation (1), we obtain

$$\overrightarrow{P} - \overrightarrow{Q} = \frac{1}{2}\overrightarrow{B} - \frac{1}{2}\overrightarrow{A} = x\overrightarrow{A} - y\overrightarrow{B} .$$

By the unique linear combination theorem (Theorem 3-5), $x = -\frac{1}{2}$ and $y = -\frac{1}{2}$. The geometric interpretation of this discovery is that O is a trisection point of $\overline{AP}$ and $\overline{BQ}$. If we substitute these values in Equations (1) and (2) and add, we obtain

$$\overrightarrow{P} + \overrightarrow{Q} = \frac{1}{2}\overrightarrow{A} + \frac{1}{2}\overrightarrow{B} + \overrightarrow{C} = -\frac{1}{2}\overrightarrow{A} - \frac{1}{2}\overrightarrow{B} .$$

Since $\overrightarrow{R} = \frac{1}{2}(\overrightarrow{A} + \overrightarrow{B})$, the second two members of this equality become

$$\overrightarrow{R} + \overrightarrow{C} = -\overrightarrow{R} \quad \text{or} \quad \overrightarrow{R} = -\frac{1}{2}\overrightarrow{C} .$$

Thus, R and C are collinear, O is on $\overline{CR}$, and O is a point of trisection of $\overline{CR}$.

If we choose another point as origin and let G be the point of intersection of the medians, the Point of Division Theorem permits us to write

$$\overrightarrow{G} = \frac{1}{3}\overrightarrow{A} + \frac{2}{3}\overrightarrow{P} ,$$

or

$$\overrightarrow{G} = \frac{1}{3}\overrightarrow{A} + \frac{2}{3}\left(\frac{1}{2}\overrightarrow{B} + \frac{1}{2}\overrightarrow{C}\right) = \frac{1}{3}\overrightarrow{A} + \frac{1}{3}\overrightarrow{B} + \frac{1}{3}\overrightarrow{C} = \frac{1}{3}(\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}) .$$

We have not only solved the problem, but also have represented the point of concurrency by the vector $\frac{1}{3}(\vec{A} + \vec{B} + \vec{C})$. This point is called the centroid of the triangle and has an important property connected with the idea of the center of gravity of a physical object. If a thin uniform sheet (such as cardboard) is cut in the shape of the triangle, it can be balanced on a pencil point placed at the point corresponding to the centroid.

Example 4. Show that the bisector of an angle of a triangle divides the opposite side into segments whose lengths are proportional to the lengths of the adjacent sides.

Solution. Let $\overline{PT}$ bisect $\angle QPR$, and let the vector from P to Q be represented by $\vec{a}$, the vector from P to T by $\vec{b}$, and the vector from P to R by $\vec{c}$, as shown in Figure 4-13. We are to show that

$$\frac{d(R,T)}{d(T,Q)} = \frac{d(P,R)}{d(P,Q)} .$$

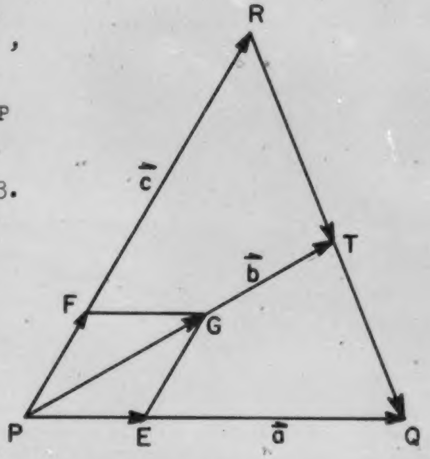


Figure 4-13

This problem involving an angle bisector affords us an opportunity to demonstrate the use of unit vectors in a solution. A vector which bisects the angle between $\vec{a}$ and $\vec{c}$ must lie along the diagonal of a rhombus whose adjacent sides lie along $\vec{a}$ and $\vec{c}$. We employ unit vectors to accomplish this result.

Any vector along $\vec{a}$ can be represented as a scalar multiple of $\vec{a}$. In particular, the unit vector along $\vec{a}$ can be represented by $\frac{1}{|\vec{a}|}\vec{a}$ or $\frac{\vec{a}}{|\vec{a}|}$.

Then the vector from P to E , $\frac{\vec{a}}{|\vec{a}|}$, and the vector from P to F , $\frac{\vec{c}}{|\vec{c}|}$, determine a rhombus whose diagonal $\overline{PG}$ bisects the angle determined by $\vec{a}$ and $\vec{c}$. The vector from P to G is then $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}$, and any vector along it, say from P to T , can be represented by a scalar multiple, $k\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right)$.

Now suppose r is the ratio $\frac{d(R,T)}{d(R,Q)}$. Since the vector from R to Q is $(\vec{a} - \vec{c})$, the vector from R to T may be expressed as $r(\vec{a} - \vec{c})$, and that from T to Q by $(1 - r)(\vec{a} - \vec{c})$. We may write

$$\vec{b} = \vec{c} + r(\vec{a} - \vec{c})$$

and obtain $k\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right) = \vec{c} + r(\vec{a} - \vec{c})$, or $\frac{k}{|\vec{a}|}\vec{a} + \frac{k}{|\vec{c}|}\vec{c} = r\vec{a} + (1 - r)\vec{c}$.

Equating the corresponding coefficients, we have

$$\frac{k}{|\vec{a}|} = r \quad \text{and} \quad \frac{k}{|\vec{c}|} = 1 - r .$$

It follows that

$$\frac{r}{1 - r} = \frac{|\vec{c}|}{|\vec{a}|} ;$$

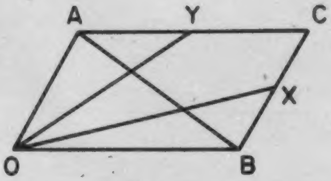
hence,

$$\frac{d(R,T)}{d(T,Q)} = \frac{d(P,R)}{d(P,Q)} .$$

Exercises 4-3

1. Give a vector proof that the diagonals of a parallelogram bisect each other.

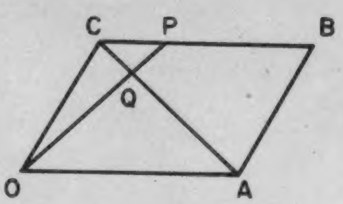
2. Prove by using vectors that a line segment which joins one vertex of a parallelogram to the midpoint of an opposite side passes through a point of trisection of a diagonal. ($\overline{AB}$ in the figure.) Prove also that the diagonal $\overline{AB}$ passes through points of trisection of $\overline{OX}$ and $\overline{OY}$.



3. Rework Example 3 for the case in which the origin is selected to be the point A . Does this choice of origin simplify the proof?

4. In parallelogram OABC , $\overline{OP}$ intersects $\overline{AC}$ at Q .

If $\frac{d(C,P)}{d(C,B)} = \frac{1}{r}$, show that $\frac{d(C,Q)}{d(C,A)} = \frac{1}{r+1}$.



Exercises 5 to 10 are theorems from plane geometry which you are to prove by the vector methods illustrated in the examples of this section.

5. If two medians of a triangle have equal length, then the triangle is isosceles.

6. The median to the base of an isosceles triangle is perpendicular to the base.

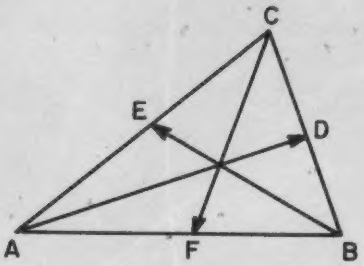
7. The line segments joining the midpoints of the opposite sides of a quadrilateral bisect each other.

8. The line segment joining the midpoints of two sides of a triangle is parallel to the third side and has length equal to one-half the length of the third side.

9. An angle inscribed in a semicircle is a right angle.

10. The bisectors of a pair of adjacent supplementary angles form a right angle.

11. D, E, and F are midpoints of $\triangle ABC$, as shown. Let the vector from A to D be $\overrightarrow{a}$, the vector from B to E be $\overrightarrow{b}$, the vector from C to F be $\overrightarrow{c}$. Prove that $\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c} = \overrightarrow{0}$.



4-4. Proofs Using Polar Coordinates.

Polar coordinates are useful in many applications, particularly if the problems involve rotations or trigonometric functions.

The following example from trigonometry illustrates one such use.

Example 1. Show that $\cos(\beta - \alpha) = \cos\beta \cos\alpha + \sin\beta \sin\alpha$.

Let $\angle\alpha$ and $\angle\beta$ be as shown in Figure 4-14. We select points B and C on the respective terminal sides of the angles and let $d(B,C) = a$, $d(A,C) = b$, and $d(A,B) = c$. The distance formula tells us that

$$(1) \quad a^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 .$$

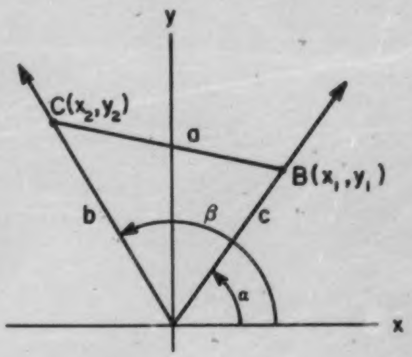


Figure 4-14

Now if we convert from rectangular to polar coordinates as outlined in Section 2-5, Equation (1) becomes

$$a^2 = (b\cos\beta - c\cos\alpha)^2 + (b\cdot\sin\beta - c\sin\alpha)^2 .$$

Expanding the right member and applying the identity $\sin^2\theta + \cos^2\theta = 1$, we obtain

$$(2) \quad a^2 = b^2 + c^2 - 2bc(\cos\beta \cos\alpha + \sin\beta \sin\alpha) .$$

Noting that the measure of $\angle BAC = \beta - \alpha$ and comparing Equation (2) with the Law of Cosines for $\triangle ABC$, we see that

$$\cos(\beta - \alpha) = \cos\beta \cos\alpha + \sin\beta \sin\alpha .$$

As for the next example, it is unlikely that anyone would choose this kind of proof when other proofs are available, but nevertheless, it may be instructive to look at one demonstration of a simple geometric proposition using polar coordinates.

Example 2. Prove that the median to the base of an isosceles triangle bisects the vertex angle.

Consider Figure 4-15, in which $\overline{AC} \cong \overline{BC}$. In order to describe the angles in question, we let C be the pole. We also let D, the midpoint of $\overline{AB}$, lie on the polar axis. Without loss of generality, we have $A = (r, \alpha)$, $B = (r, \beta)$. We must prove $\alpha = -\beta$.

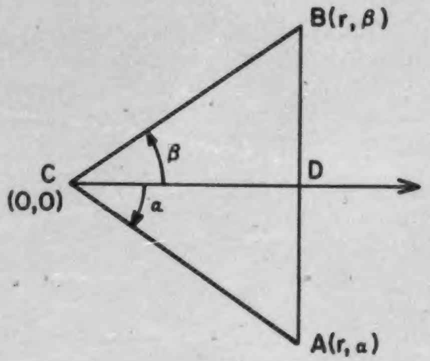


Figure 4-15

To simplify the notation we shall let $d(C,D) = f$ and $d(A,D) = d(B,D) = g$. Applying the Law of Cosines, we have,

in $\triangle BCD$, $$g^2 = r^2 + f^2 - 2rf \cos \beta ,$$

and in $\triangle ACD$, $$g^2 = r^2 + f^2 - 2rf \cos \alpha .$$

We see then that $\cos \alpha = \cos \beta$. Since $0 < \alpha < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \beta < 0$, this implies $\alpha = -\beta$.

4-5. Choice of Method of Proof.

It is time we paused to survey the variety of problem-solving tools which are now at our disposal. We have a choice of three basic systems --rectangular coordinates, polar coordinates, and vectors; within each system we have different representations to suit different purposes. But the question uppermost in your mind at the moment probably is, "How do I decide which method is the best one to use?"

The question does not have a simple answer. Some problems are best worked by one particular method, other problems seem to be approachable by any of these methods, and some problems appear to be impossible regardless of what we try.

However, there are certain guidelines which may help us.

(1) Try to decide upon a coordinate system which is appropriate to the problem. Think over what is known about the problem, or what is to be proved, or what kind of answer is required.

  (a) Distances between points, slopes of lines, and midpoints of segments are easily handled in rectangular coordinates; therefore, when these ideas are present, you should try to fit rectangular coordinate axes to the problem.

  (b) If the problem involves angular motion or circular functions, it would be wise to look at the possibilities of polar forms.

  (c) Vectors are quite versatile and fit a wide range of conditions. Concurrence, parallelism, and perpendicularity of lines, as well as problems of physical forces, are situations which might lead you to choose a vector approach.

(2) Make a drawing relating the known facts of the problem to your choice of method. Much time and effort may be saved by a reasonably accurate drawing. This not only helps to relate the parts of the problem, but it serves as a check on the calculated results.

(3) Choose coordinates or vectors so as to simplify the algebra. Take advantage of all the given information at this stage, but be careful that you maintain generality where it is required.

(4) Watch for opportunities to use parametric representations. This may be something new to you, but you will observe frequent cases in succeeding chapters in which this special method will simplify troublesome problems.

(5) Work many, many problems. It also will help if you try to solve a given problem in several different ways. In this area of mathematics, experience is probably the most valuable asset. Sometimes a choice of method can be explained only on the basis of experience.

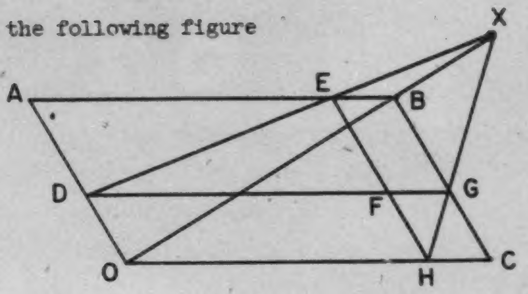
(6) After you have completed your solution to a problem, it is wise to look back over your work. You may see an unnecessary step you can eliminate, an unwarranted assumption you should justify, or a general tightening up you may accomplish. In any case, you gain a new perspective on your work which increases your understanding and appreciation of what you have done.

Review Exercises

For Exercises 1 to 10, first choose a coordinate system which you think is appropriate for each theorem, and then prove the theorem accordingly.

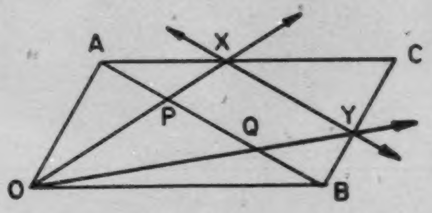
1. The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.
2. The locus of the vertex of a right angle, the sides of which pass through two fixed points, is a circle.
3. The diagonals of a rectangle have equal length.
4. Show that the sum of the squares of the lengths of the sides of a parallelogram is equal to the sum of the squares of the lengths of its diagonals.
5. The line segments joining in order the midpoints of the successive sides of an isosceles trapezoid form a rhombus.
6. The line segment joining the midpoints of the diagonals of a trapezoid is parallel to the bases and has length equal to one-half the difference of the lengths of the bases.
7. If lines are drawn through a pair of opposite vertices of a parallelogram and through the midpoints of a pair of opposite sides in such a way that the lines intersect one of the diagonals in distinct points, the lines are parallel and the diagonal is trisected.
8. The perpendicular bisectors of the sides of a triangle are concurrent in a point that is equidistant from the three vertices of the triangle.
9. If two sides of a triangle are divided in the same ratio, the line segment joining the points of division is parallel to the third side and is in the same ratio to it.
10. Show that the vector joining the midpoints of two opposite sides of a vector quadrilateral is equal to half the vector sum of the other two sides.

12. In the following figure



OABC , DAEF , and HFGC are each parallelograms. Prove that the respective diagonals of the parallelograms $\overline{OB}$, $\overline{DE}$, and $\overline{HG}$, extended as necessary, meet in a single point X .

13. In parallelogram OACB , let P and Q be points on diagonal $\overline{AB}$ such that $d(A,P) = d(B,Q)$. Let $\overrightarrow{OP}$ intersect $\overleftrightarrow{AC}$ at X , and let $\overrightarrow{OQ}$ intersect $\overleftrightarrow{BC}$ at Y . Show that $\overleftrightarrow{XY} \parallel \overleftrightarrow{AB}$.

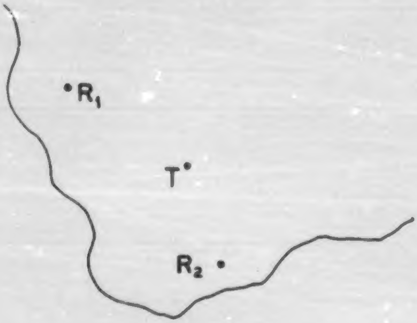


14. Prove that the sum of the squares of the lengths of the sides of a quadrilateral exceeds the sum of the squares of the lengths of its diagonals by 4 times the square of the length of the line segment that joins the midpoints of the diagonals.

15. A band of pirates buried their treasure on an island. They chose a spot at which to bury it in the following manner: Near the shore there were two large rocks and a large pine tree. One pirate started out from one rock along a line at right angles to the line between this rock and the tree. He marched a distance equal to the distance between this rock and the tree. Another pirate started out from the second rock along a line at right angles to the line between this second rock and the tree and marched a distance equal to the distance between this rock and the tree.

The rest of the band of pirates then found the spot midway between these two and there buried the treasure.

Many years later, these directions came to light and a party of treasure-seekers sailed off to find the treasure. When they reached the island, they found the two rocks with no difficulty. But the tree had long since disappeared, so they did not know how to proceed. All seemed lost till the cabin boy, who had just finished his freshman year at Yale, spoke up. Remembering the analytic geometry he had studied, he calculated where the treasure must be, and a short spell of digging proved him correct. How did he do it?



Chapter 5

GRAPHS AND THEIR EQUATIONS

5-1. Introduction

In Section 2-2 we discussed sets of points and their analytic representations. The relation between the two is at the heart of analytic geometry, and we shall review the fundamental notions briefly here. We confine the discussion to the plane, but the extension to space is immediate. The sets of points will frequently be the geometric figures we met earlier, and the analytic representations will usually be given in algebraic or trigonometric forms that we have met before. We propose to relate these ideas with the hope that your competence and appreciation for their use will continue to grow.

Let S be a set of points in a plane with a rectangular coordinate system. Let $s(x,y)$ be an open sentence involving two variables. Let S consist of those points (a,b) of the plane such that $s(a,b)$ is true. Then we say S is the *locus* (or *graph*) of the condition $s(x,y)$, and $s(x,y)$ is a *condition* for the set S. The plural of "locus" is "loci". (It is pronounced as though it were spelled "low-sigh". The rectangular coordinate system in the plane could be replaced by any other coordinate system appropriate to the problem and to the space in which we are working. The choice of a coordinate system determines the "language" in which the open sentence is stated. We shall often be concerned with the limitations of a particular language, and the details of the translation from one language to another.

Some of you may be used to a different way of talking about the matter. In the SMSG *Geometry* there is a discussion of characterizations of sets. A condition is said to characterize a set if every point in the set satisfies the condition and every point that satisfies the condition is in the set. The conditions we are chiefly interested in here are analytic conditions (conditions on the coordinates of points), whereas in *Geometry* the conditions were stated in geometric terms.

5-2. Conditions for Loci or Graphs, and Graphs of Conditions

The discussion above is quite general, but in practice the conditions that matter most are equations and inequalities. For example, we define the graph of an equation (inequality) in x and y to be the set of points whose coordinates satisfy the equation (inequality). Thus the locus of the equation $x^2 + y^2 = 4$ is the circle with center $(0,0)$ and radius 2, while the locus of the inequality $xy < 0$ is the set of points in the second quadrant or in the fourth quadrant. Using set notation these two loci can be expressed as follows:

$$\{P = (x,y) : x^2 + y^2 = 4\} ,$$

$$\{P = (x,y) : xy < 0\} .$$

Using the same notation we can express the loci of the equation $f(x,y) = 0$, and the inequality $g(x,y) > 0$ as follows:

$$\{P = (x,y) : f(x,y) = 0\} ,$$

$$\{P = (x,y) : g(x,y) > 0\} .$$

We now take up the problem of finding an analytic condition for a set of points in a plane. There is no routine procedure for doing this, but the following advice may be useful.

First a word about the choice of coordinate systems. When the terms of the problem leave you free, think carefully about the coordinate system to use. Some curves with complicated equations in rectangular coordinates have nice parametric representations. An equation in rectangular coordinates for a certain curve may be simpler than it is otherwise if a coordinate axis is an axis of symmetry. A circle of radius 3 has a simple equation in rectangular coordinates if its center is made the origin, a still simpler equation in polar coordinates if its center is chosen as the pole.

Following common usage we will use x and y for rectangular coordinates, and r and θ for polar coordinates. We will also assume in each case, unless otherwise specified, suitable choices of axes and units. Only with these assumptions may we speak about "the" locus of an equation. Without such assumptions an equation may have several quite different graphs, depending on our choices of coordinate systems. These matters will be considered more fully later, particularly in Chapter 6.

After choosing a coordinate system we can attack the problem. We start with a given set of points. These points are not given to us in a basket but instead are determined by some geometric condition. We are looking for an equivalent condition in terms of the coordinates of points. Let us look at what we do in several examples.

Example 1. We describe certain sets of points of the plane. You are asked to give analytic description of each set.

(a) All the points of the x-axis.

Solution. $\{P = (x,y) : y = 0\}$.

(b) All the points above the x-axis.

Solution. $\{P = (x,y) : y > 0\}$.

(c) All the points of the plane except those on either axis.

Solution. $\{P = (x,y) : xy \neq 0\}$.

(d) The midpoints of all line segments in the first quadrant which, with the coordinate axes, form a triangle whose area has a measure of 12 square units.

Solution. If $P = (x,y)$ is one such point, the endpoints of its segment have coordinates $(2x,0)$ and $(0,2y)$. The triangular region will then have area $\frac{1}{2}(2x)(2y)$, which must equal 12 . We have the simpler equivalent relationship $xy = 6$. The graph of this relationship contains points in the first and third quadrants but we want only those with positive coordinates. Thus, our answer is $\{P = (x,y) : xy = 6 , x > 0 , y > 0\}$.

Example 2. Find an equation in rectangular coordinates of the locus of all points equidistant from two distinct points.

Solution. Let the x-axis be the line through the two points and let the origin be the midpoint of the segment determined by them. Then the two points are $(a,0)$ and $(-a,0)$. Let (x,y) be any point in the plane. Then the distances to (x,y) from $(a,0)$ and $(-a,0)$ are $\sqrt{(x - a)^2 + y^2}$ and $\sqrt{(x + a)^2 + y^2}$, respectively. The point (x,y) belongs to our locus if and only if these two distances are equal, that is, if and only if

$$\sqrt{(x + a)^2 + y^2} = \sqrt{(x - a)^2 + y^2} \quad (1)$$

Thus (1) is an equation of the locus. (1) is, of course, not the simplest possible equation for the locus. What is, and how can you get it from (1) ?

Example 3. We present some analytic descriptions of sets of points of the plane. Describe these sets in ordinary English.

(a) $\{P = (r,\theta) : r > 5\}$.

Solution. All points outside a circle whose center is at the pole and whose radius is 5 .

(b) $\{P = (x,y) : |x - 3| = 7\}$.

Solution. All the points on two parallel lines. These lines are parallel to the line $x = 3$, and lie one on each side of it and 7 units away.

(c) $\{P = (x,y) : xy + 2x - y > 2\}$.

Solution. This inequality may be written $xy + 2x - y - 2 > 0$, or $(x - 1)(y + 2) > 0$. This statement will be true for values of x and y such that either:

$$x - 1 > 0 \text{ and } y + 2 > 0 \text{ , or } x - 1 < 0 \text{ and } y + 2 < 0 \text{ ;}$$

that is if either:

$$x > 1 \text{ and } y > -2 \text{ , or } x < 1 \text{ and } y > -2 \text{ .}$$

The points we want lie in two "quadrants", as indicated in Figure 5-1. The graph does not include the boundaries of the regions. How could you change the analytic descriptions of the set to include these boundaries?

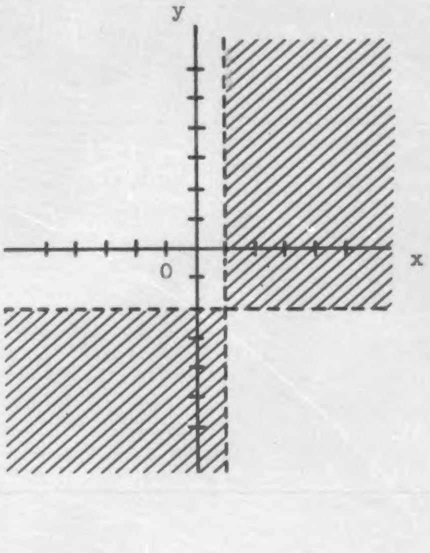


Figure 5-1

(d) $\{P(x,y) : |x + 1| < 3 \text{ and } |y + 1| \le 4\}$.

Solution. All the points of a rectangular region, with center at the point $(-1,-1)$. The region is 6 units wide and does not include the vertical boundaries; it is 8 units high and does include the horizontal boundaries. It is pictured in Figure 5-2. We note that the corners of the region are not points of the graph.

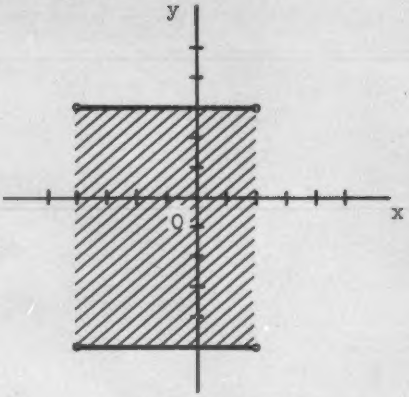


Figure 5-2

(e) $\{P = (r,\theta) : |r - 5.0| < .1\}$.

Solution. The set of points of the annular region between two concentric circles centered at the pole. The inner circle has radius 4.9 and the outer circle has radius 5.1 , but neither circle is part of the locus, which is illustrated in Figure 5-3.

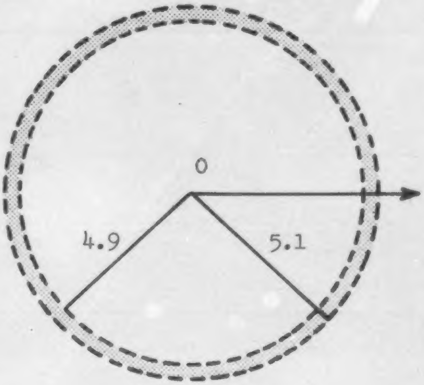


Figure 5-3

We have been using set notation because we wanted to be perfectly clear. Hereafter we shall be less formal. We might state the problem of Exercise 3(e): Describe and draw the graph of $|r - 5.0| < .1$.

Example 4. Find an equation in rectangular coordinates for the locus of all points which are equidistant from a given point F and a given line L .

Solution. The geometric condition for the locus defines a parabola, whose equation we now derive from the condition. With this in mind we let the line through F perpendicular to L be the y-axis, with the origin at the midpoint of the segment determined by F and the point where the perpendicular intersects L . (If F is in L , we pick F as the origin and leave the further details in this case as an exercise.) Finally, we let the y-coordinate of F be $\frac{p}{2}$, where $p \geq 0$. Then $F = (0,\frac{p}{2})$ and L is the line $y = -\frac{p}{2}$.

Let $P = (x,y)$ be an arbitrary point in the plane. Then the things talked about in the geometric condition are the distances from P to F and to L . Using the distance formula we find that the first of these is $\sqrt{x^2 + (y - \frac{p}{2})^2}$. The second is $|y + \frac{p}{2}|$. The geometric condition says these two distances are to be equal. Hence

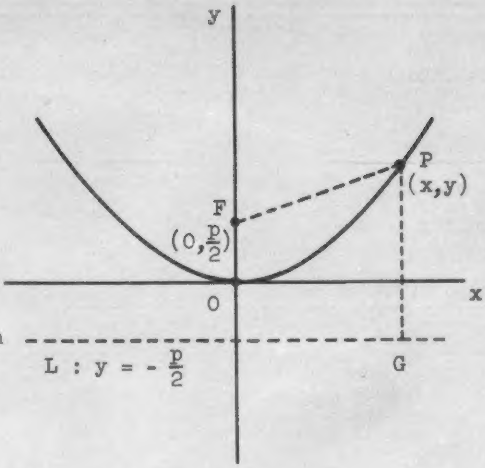


Figure 5-4

$$\sqrt{x^2 + (y - \tfrac{p}{2})^2} = |y + \tfrac{p}{2}| \tag{2}$$

is an equation for the locus. This is a complete solution of the original problem, but a simpler equation can be found. If we square both members of (2) and combine terms, we get the equation

$$x^2 = 2py . \tag{3}$$

There remains the question of whether (2) and (3) are equivalent. The only operation we have performed which might have caused trouble was the squaring of both sides. But any point on the locus of (2) is on the locus of the equation obtained by squaring both members of (2) , and hence on the locus of (3) . That the reverse is also true can be shown most simply by considering a more general problem. Let (a,b) be a point on the locus of $\big(f(x,y)\big)^2 = \big(g(x,y)\big)^2$, so that $\big(f(a,b)\big)^2 = \big(g(a,b)\big)^2$. Then $f(a,b) = \pm g(a,b)$. Now suppose, further, that if (x,y) is in the domains of f and g , then $f(x,y) \geq 0$ and $g(x,y) \geq 0$. We cannot have $f(a,b) = -g(a,b)$ unless both are zero, and hence $f(a,b) = g(a,b)$. Thus $\big(f(x,y)\big)^2 = \big(g(x,y)\big)^2$ and $f(x,y) = g(x,y)$ are equivalent equations. This result settles our question for us, since both members of (2) are non-negative for all x and y .

Example 5. A Coast Guard cutter, searching for a boat in distress, travels in a path with the property that the distance (in miles) of the cutter from its starting point, O , is equal to the radian measure of the angle generated by the ray from O to the cutter. Find an equation of the path in a suitable coordinate system. (Assume the surface of the ocean is a plane.)

Solution. The description of the path suggests that we should use polar coordinates, with O as pole and the polar axis in the direction in which the cutter is heading when it starts its search. If we do this we get immediately the function defined by the equation $r = \theta$. (By choosing the positive direction of rotation properly we can make θ positive.)

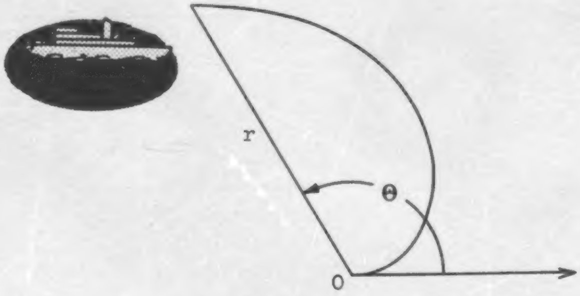


Figure 5-5

The path is a spiral.

If we use rectangular coordinates we get a much more complicated equation. Furthermore, no matter how we choose the axes, the equation does not define a function. Can you explain why not?

Related Polar Equations. In writing an analytic description of a set of points we may use to our advantage the freedom we have in choosing the type of coordinate system, the placement of the axes, and the units. In the case of polar coordinates there is an ambiguity imposed on us by the fact that each point now has infinitely many pairs of coordinates. This makes some matters easy, and some difficult. If a moving point traces and retraces its path in a recurrent pattern, a polar equation for the locus can represent this pattern, since (r,θ) and $(r,\theta + 2\pi n)$ are, for integral values of n , coordinates for the same point. On the other hand, since (r,θ) and $(-r,\theta + \pi)$ are also coordinates for the same point, we cannot avoid a certain ambiguity in writing equations of loci in polar coordinates. A point (r_1,θ_1) on the curve represented by the equation $r = f(\theta)$ also has the coordinates

$(-r_1, \theta_1 + \pi)$. If we substitute the latter coordinates in the equation we obtain the equation $-r_1 = f(\theta_1 + \pi)$ which may be written $r_1 = -f(\theta_1 + \pi)$. That is, every point of the curve represented by $r = f(\theta)$ is at the same time a point of the curve represented by $r = -f(\theta + \pi)$. We will call these equations,

$$\begin{cases} r = f(\theta) , \\ r = -f(\theta + \pi) , \end{cases}$$

related polar equations for the curve. In some cases these related polar equations are quite different in appearance and it takes some experience to recognize that they represent the same curve. On the other hand the related polar equations may be identical.

Example 6. The related equation for $r = 5 \sin \theta$ is $r = -5 \sin(\theta + \pi) = -5(-\sin \theta) = 5 \sin \theta$, and is the same as the original equation.

Example 7. The related equation for $r = 3 \tan \theta$ is $r = -3 \tan(\theta + \pi) = -3 \tan \theta$, and is different from the original equation.

Example 8. The related equation for $r = 3(1 + \sin \theta)$ is $r = -3(1 + \sin(\theta + \pi)) = -3(1 - \sin \theta) = 3(\sin \theta - 1)$, and is different from the original equation.

Example 9. The related equation for $r = 5$ is $r = -5$, and is different from the original equation.

Because the correspondences between points and their polar coordinates and between sets of points and their representations in polar coordinates are not unique, we must define the graph of a polar equation to be not the set of points whose coordinates satisfy that equation but rather the set of points each of which has some pair of coordinates that satisfy the equation.

Exercises 5-2

For each of the following, write an equation or statement of inequality of the locus of a point which satisfies the stated condition. Use the coordinate system you think appropriate if one is not specified. If you use polar coordinates, give the pair of related equations in each case.

1. A point 3 units above the x-axis.
2. A point 5 units to the left of the y-axis.
3. A point equidistant from the x- and y-axes.
4. A point twice as far from the x-axis as it is from the y-axis.
5. A point a units from the origin.
6. A point a units from the point $(3,-2)$.
7. A point equidistant from $(3,0)$ and $(-5,0)$.
8. A point equidistant from $(2,3)$ and $(5,-4)$.
9. A point equidistant from the lines with equations $x + y - 2 = 0$ and $x + 2y + 2 = 0$.
10. A point whose distance from the line with equation $x + 2 = 0$ is equal to its distance from the point $(2,0)$.
11. A point whose distance from the line with equation $2x + y + 2 = 0$ is equal to its distance from the point $(2,-1)$.
12. A point the sum of whose distances from the points $(4,0)$ and $(-4,0)$ is 10.
13. A point the difference of whose distances from the points $(4,0)$ and $(-4,0)$ is 6.
14. A point the ratio of whose distances from the lines $2x + y - 4 = 0$ and $3x - y + 1 = 0$ is 2 to 3.
15. A point that is contained in the line through the points $(-1,2)$ and $(5,7)$.
16. A point, the product of whose distances from two fixed points is a constant. (This locus is called Cassini's Oval; it was studied by Giovanni Domenico Cassini in the late seventeenth century in connection with the motions of the earth and the sun.)
17. A point within 3 units distance from the x-axis.
18. A point at least 5 units distant from the origin.
19. A point no more than 1 unit from the y-axis.
20. A point no more than 2 units from $(1,3)$.
21. A point no nearer to the origin than it is to the point $(0,5)$.
22. A point no nearer to the origin than it is to the line $y = 4$.

23. A point nearer to the origin than to any point on the line $x = 10$.

24. A point between the lines $x = 6$, $x = -6$.

25. A point within a circle with its center at the origin, if the radius is "8 inches $\pm$ 1 % ." (Note: This notation, frequently seen in drawings and applications, means here that the radius must be at least 7.92 inches long, and at most 8.08 inches long. We sometimes say that there is a "tolerance" of 1 % of the stated dimension.)

5-3. Parametric Representation.

In describing physical phenomena we customarily simplify matters; for example, a car on the road becomes a point on the line. In describing any motion it is convenient to say when, after some given instant, a particular event occurs. This is indicated by a value of the variable, t . If the motion takes place in two or three dimensions its analysis may be made easier by considering one dimension at a time. With a rectangular coordinate system we may then describe that part of the motion parallel to the x-axis (the x-component) by indicating how it alone changes with respect to time, say $x = f_1(t)$. Similarly we may have $y = f_2(t)$. Such a set of equations, in which the two components of the motion, that is, the values of the two variables x and y are given in terms of a third variable, t , is an example of what is called a parametric representation of the motion. It is interesting to note that the tracking of satellites is actually done in this way.

Example 1. Two students observe the motion of a ball rolling down a tilted plane. The plane has been coordinatized as indicated. In this illustration, as in many physical problems, the variable t , representing time elapsed since a given instant, is used as a parameter or auxiliary variable. The use of a parameter is often of great value in simplifying the presentation and solution of physical problems. In some problems it may be useful to use two, or even more, parameters.

One student finds that with suitable units he can describe the motion relative to the y-axis with the equation $y = 3t^2$.

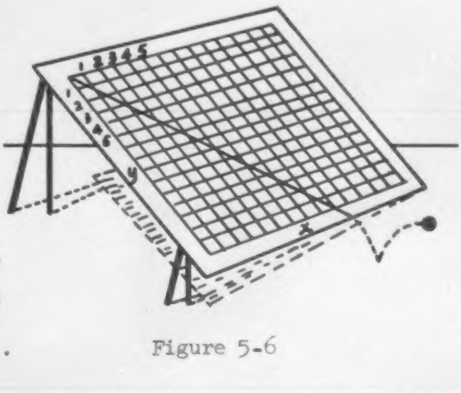


Figure 5-6

He may have come to this conclusion by noting, with the use of a stop-watch, the y-coordinates of the points on the lines parallel to the x-axis crossed by the rolling ball in successive seconds. The other student, using the lines parallel to the y-axis in a similar way, finds that he can describe the motion relative to the x-axis with the equation $x = 2t^2$. These are the parametric equations of the motion. If we want to express y in terms of x, we may eliminate t between these two equations and obtain $y = \frac{3}{2}x$. Since t is a measure of elapsed time it is nonnegative, hence x and y are also nonnegative. Therefore, the graph on the xy-plane will be a ray of the line whose equation may be written $y = \frac{3}{2}x$.

Example 2. A plane, flying at 120 miles per hour at an altitude of 5000 feet, drops a package to the ground. Assume that the package remains in one vertical plane as it falls, and, neglecting air resistance, determine its path to the ground.

Solution. We must assume certain conditions. If, at the moment of its release, the package is moving forward at 120 mph (= 176 ft. per sec.), then it will continue to do so at the same rate, whatever its vertical motion may be. Under the stated conditions we assume that its vertical motion is described by the formula $s = \frac{1}{2}gt^2$, where t represents the elapsed time in seconds, g is the gravitational acceleration in feet per second per second (which we shall approximate as 32), and s is the number of feet of free fall.

We now coordinatize the vertical plane, taking the point of release as the origin. The positive sense of the x-axis indicates forward motion, and the positive sense of the y-axis indicates downward motion.

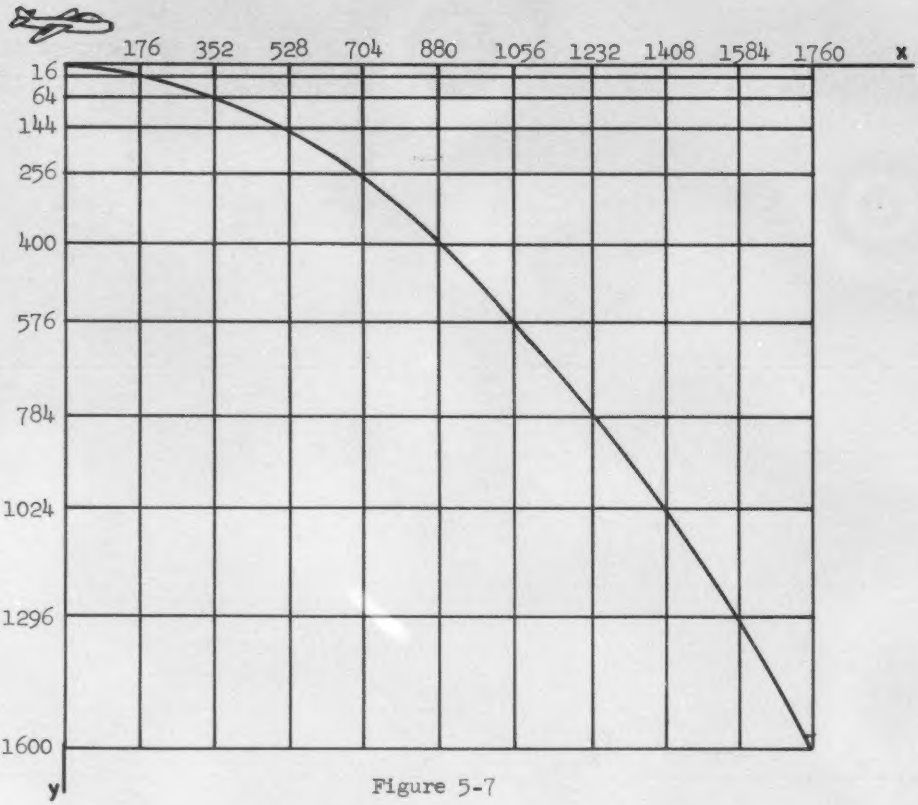


Figure 5-7

Note that the grid on which the locus is drawn has been presented in a non-standard way, to make the diagram easier to interpret. As the package moves forward in space the corresponding point on the graph moves right and crosses successive vertical lines in successive seconds. The vertical lines are equally spaced because the horizontal motion is uniform: $x = 176t$. As the package falls the corresponding point on the graph moves down on the page, crossing successive horizontal lines in successive seconds. The horizontal lines are not equally spaced because the vertical motion is not uniform, but accelerated. The spacing was determined by successive values of t in the formula $y = 16t^2$. The scale is the same on both axes, thus the diagram is not only a graph of our locus, but also a picture of the actual path.

If we had plotted points on a different grid, say the one to the right, in which the horizontal scale is different from the vertical scale, then the graph would still be an accurate representation of the relationships among the variables, but it would not be an accurate representation of the path. Since we use the word path here in a special way, we define it to be the set of positions actually occupied by a real object as it moves in real space. Clearly, a path may be represented by a curve in a great number of ways by different choices of coordinate systems.

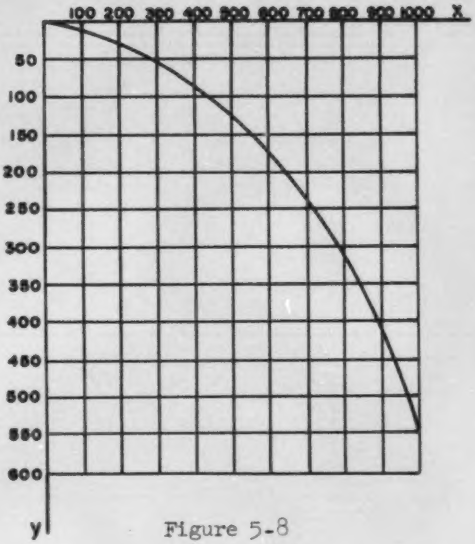


Figure 5-8

In many physical problems we are concerned with the relative positions of objects as they travel on their respective paths. If the bat is to hit the ball, it is not enough for their paths to cross, they must be at the crossing point at the same time. Ships' paths may cross safely, but a collision course would bring them to the same point at the same moment. The captains of two ships at sea are concerned with when and where the ships are closest to each other. When we must consider time and position along a path, we need some relationship involving these quantities. These are most readily presented in parametric form.

Exercises 5-3

1. Refer to Example 1 and make a chart like the one below, showing the x and y coordinates for integral values of t from $t = 0$ to $t = 10$.

t	0	1	2	3	4	5	6	7	8	9	10
x											
y											

2. Make a similar chart for Example 2 of this section.

3. Write parametric equations for the position of a point $P = (x,y)$ which starts on the y-axis and moves across the plane at the rate of 5 units a second, and remains always 2 units above the x-axis.

4. Write parametric equations for the position of a point $P = (x,y)$ which starts on the x-axis and moves uniformly on the plane at the rate of 2 units a second, and remains always 6 units to the left of the y-axis.

5. Write parametric equations for the position of a point $P = (x,y)$ which starts at the origin, goes through the point $(3,4)$ ten seconds later, and continues to move uniformly along line $\overleftrightarrow{OP}$ at that same rate across the plane. Find rectangular equations for its locus.

6. Write parametric equations for the position of a point $P = (x,y)$ which moves uniformly along a line across the plane, and takes 5 seconds to go from $(-6,1)$ to $(1,25)$.

7. Parametric equations for the path of a point $P = (x,y)$ are $x = t$, $y = t^2$, where t indicates time in seconds. Discuss the motion of the point in the first five seconds. Make an estimate, correct to the nearest .1 unit, of the distance traveled in that time.

8. A point $P = (x,y)$ travels along the line represented by $4x - 3y + 2 = 0$ at the uniform rate of 10 units per second and passes through $(1,2)$ when $t = 3$. Write parametric equations for its position at any time t. Find its position when $t = 0$; when $t = 10$.

9. A point $P = (x,y)$ travels along the line represented by $2x + 3y - 6 = 0$ at a uniform rate of 5 units per second and crosses the x-axis at the time $t = 0$. Write parametric equations for its position at any time t.

10. A point $P = (x,y)$ moves uniformly on a line across the plane. It goes through (a,b) at time t_0, and (c,d) at time t_1. Write parametric equations for its position at any time t.

11. A point is moving along the x-axis, its position at time t (sec) given by $x = \cos t$. Before you do any computation try to describe the way the point moves. The cosine function is frequently associated with angles and rotation, but there is no such motion here. We must now use the cosine as a particular real number function, whose values, for the domain $0 \le x \le 1.80$ are given in Table II. The heading "radian measure" for that table indicates the most frequent but by no means the only use for these trigonometric functions. Make a table for the positions of the point for the first 10 seconds, at one second intervals. How would you find the position of the point at the end of one minute? one hour?

12. The vertical position of a point is given by $y = 500 - 16t^2$ where y represents altitude in feet and t elapsed time in seconds. Before you do any computation try to describe the motion of the point. Do you know any physical interpretation of this motion? Make a table of the position of the point, at one second intervals, for the first 10 seconds.

13. Refer to the previous exercise, and answer the same questions for the relationship $y = 120 + 64t - 16t^2$.

14. Refer to Exercise 11, and answer the same questions for the relationsnip $x = 4 \sin 2t$.

15. Refer to Exercise 11, and answer the same questions for the relationship $x = 2 - \cos t$.

16. If the points of Exercises 11 and 15 were on the same x-axis, find a time and place at which they meet.

5-4. Parametric Equations of the Circle and the Ellipse.

In many physical situations an important role is played by a fixed reference point, such as a source of light or radiation or a magnetic pole. The associated phenomena, sometimes called focal or radial, can be described with polar coordinates or vectors. We should use the coordinate system and parameters which seem appropriate. When rotations are involved it is usually helpful to use as a parameter, θ , the measure of the angle of rotation from a fixed initial position.

Example 1. A point moves around a circle at constant speed. Find analytic conditions for its path.

Solution: Suppose, as in the diagram, the point starts from A and moves counter-clockwise. Its position at any point P is given by the rectangular coordinates (x,y) , or the equivalents $(r \cos \theta , r \sin \theta)$; that is,

$$\begin{cases} x = r \cos \theta , \\ y = r \sin \theta . \end{cases}$$

These are parametric equations for a circle.

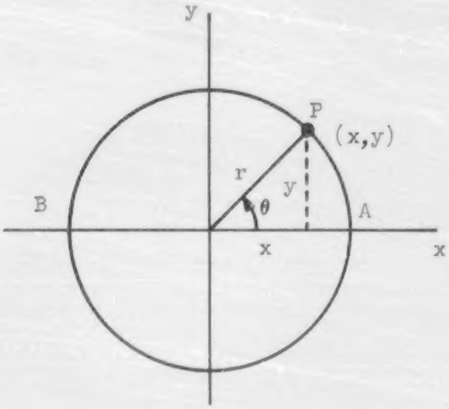


Figure 5-9

We may express the fact the point moves around the circle with constant speed by saying either that it moves along the circle at so many inches per second, or that the radius $\overline{OP}$ rotates about O at so many revolutions per minute. Of course, other units may be used. The first method of expression is important in mechanical problems involving, for example, gearing, belting, rimspeed, and so on. The second method of expressing constant speed, which concerns the amount of turning done in a unit of time, is significant in timing mechanisms such as are used in automatic washers, in electrical theory involving alternating current, which is related to the positions of a turning armature, and in the analysis of many other phenomena which are periodic, that is, which repeat in successive time intervals.

In this latter interpretation it is customary to use the Greek letter ω to represent the angular velocity, usually but not necessarily in terms of radians per unit time. Thus, if a wheel is turning at the rate of 300 revolutions per minute, it has an angular velocity of $(300)2\pi$ radians per minute, or 10π radians per second; that is, $\omega = 300(\text{rpm})$, or $\omega = 600\pi$ (radians/minute), or $\omega = 10\pi$ (radians/second).

If the point P has constant angular velocity ω, then its angular position θ is given by ωt. The parametric equations above become

$$\begin{cases} x = r\cos\omega t, \\ y = r\sin\omega t. \end{cases}$$

These are equations of the path of the point.

If we eliminate the parameter by squaring the members of each equation and adding the corresponding members of the new equations we obtain $x^2 + y^2 = r^2(\cos^2\omega t + \sin^2\omega t)$, or $x^2 + y^2 = r^2$. This represents the locus of the path in rectangular coordinates and no longer takes account of the position of the point at any particular instant.

Example 2. Two points travel on the same circle. They start at the same time from diametrically opposite positions and travel in opposite directions, the first at 2 rotations per second, the second at 3 rotations per second. Find analytic conditions for their paths, and the times and positions at which they coincide.

Solution. (Refer to Figure 5-9.) If the first point starts at $A = (r,0)$, and goes counterclockwise, its equations are

$$\begin{cases} x = r\cos 4\pi t, \\ y = r\sin 4\pi t. \end{cases}$$

If the second point starts at $B = (-r,0)$, and goes clockwise, its equations are

$$\begin{cases} x = r\cos(\pi - 6\pi t)\ , \\ y = r\sin(\pi - 6\pi t)\ . \end{cases}$$

If $t = 0$, the position of A is given by $(r\cos 0\ ,\ r\sin 0)$; therefore $A = (r,0)$, as indicated. At the same time $(t = 0)$, the position of B is given by $(r\cos\pi\ ,\ r\sin\pi)$; therefore $B = (-r,0)$, as indicated. As time elapses, the angle for the motion of A increases, while the angle for the motion of B decreases. As A and B rotate, only their angular positions are changing, and the rates of these angular displacements are 4π radians per second and -6π radians per second. At any instant the difference of these angular displacements is called their angular separation. It is customary to give this angular separation as the least angle between the respective radii to the points. Thus we use an angular separation of $\frac{\pi}{2}$ radians rather than 13.5π radians.

Since our two points start with an angular separation of π, their first meeting will occur when their angular displacements from their starting positions add to π; that is, when $4\pi t + 6\pi t = \pi$; $\therefore t = .1$ second. Successive meetings will occur after this when their additional angular displacements add to $2\pi, 4\pi, 6\pi, \ldots$, i.e., when $4\pi t + 6\pi t = 3\pi, 5\pi, 7\pi, \ldots$, i.e., when $t = .3, .5, .7, \ldots$. That is, they pass each other in .1 second, and every .2 second thereafter.

To find the corresponding positions, we need only substitute these values for t in the equations of motion. It is simplest to obtain first the successive angular positions $\theta_1, \theta_2, \ldots$, for their passing points.

If $t_1 = .1\ ,\ \theta_1 = .4\pi = 72^\circ$.

If $t_2 = .3\ ,\ \theta_2 = 1.2\pi = 216^\circ$.

If $t_3 = .5\ ,\ \theta_3 = 2\pi = 360^\circ$.

The rectangular coordinates of these positions are given, say for $r = 10$, by $P_1 = (10\cos 72^\circ\ ,\ 10\sin 72^\circ)$; $P_2 = (10\cos 216^\circ\ ,\ 10\sin 216^\circ)$; $P_3 = (10\cos 360^\circ\ ,\ 10\sin 360^\circ)\ \ldots$. These are equivalent to $P_1 = \big(10(.309)\ ,\ 10(.951)\big)$; $P_2 = \big(10(-.809)\ ,\ 10(-.588)\big)$; $P_3 = \big(10(1)\ ,\ 10(0)\big)$;

. . . . In usual rectangular form, rounded to hundredths, we have: $P_1 = (3.09, 9.51)$; $P_2 = (-8.09, -5.88)$; $P_3 = (10, 0)$; . . . .

Example 3. (Refer to Example 2, above.) Suppose, in the previous example, the points start as before but travel in the same direction, with the same rates as before. When and where do they pass?

Solution. The equations of motion are now:

$$\begin{cases} x = r\cos 4\pi t\,, \\ y = r\sin 4\pi t\,; \end{cases} \quad \text{and} \quad \begin{cases} x = r\cos(\pi + 6\pi t)\,, \\ y = r\sin(\pi + 6\pi t)\,. \end{cases}$$

The meetings (or overtakings) will take place now when the difference of their angular displacements is 2π , 4π , 6π , ... The first meeting will take place when $\pi + 6\pi t - 4\pi t = 2\pi$; that is, when $t = .5$ sec. After this, successive meetings will occur when $\pi + 6\pi t - 4\pi t = 4\pi$, 6π , 8π , . . . ; that is, when $t = 1.5$, 2.5 , 3.5 , ... To find the corresponding angular positions we proceed as in the previous problem and find $\theta_1 = 2\pi$, $\theta_2 = 6\pi$, etc.; that is, all overtakings will take place 1 second apart, at point A , starting at the end of the first half-second.

Example 4. A point is rotating uniformly on a circle of radius a , with its center at the point (b,0) . Find analytic conditions for its locus.

Solution. Suppose the uniform angular velocity, expressed in radians per second, is ω . From the hypothesis and the diagram, we have

$$\begin{cases} x = b + a\cos\theta\,, \\ y = a\sin\theta\,; \end{cases} \qquad \begin{cases} x = b + a\cos\omega t\,, \\ y = a\sin\omega t\,. \end{cases}$$

These are parametric equations for the locus. The first equations are positional only, the second equations relate these positions to time and describe the path of the point.

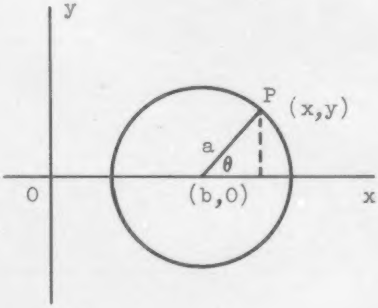


Figure 5-10

We may eliminate the parameters ω and t .

Since $$\frac{x-b}{a} = \cos\omega t\,,\ \frac{y}{a} = \sin\omega t\,,$$

therefore, $$\left(\frac{x - b}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = \cos^2 \omega t + \sin^2 \omega t = 1 \; ;$$

or $$(x - b)^2 + y^2 = a^2 \; .$$

This last equation is the one usually given in rectangular coordinates. It is an equation of the locus of the point and takes no account of its position at any particular moment.

The ellipse will be discussed in detail in Chapter 7, but we derive now its analytic representation in parametric form. We start with two concentric circles, the smallest that will enclose the ellipse, and the largest that the ellipse will enclose, as illustrated in Figure 5-11. Suppose their radii are a and b with $a > b$. We describe now a way in which a draftsman can locate as many points of the ellipse as he needs to draw a smooth curve through them. Draw any line through O, meeting the circles at A and B respectively. Through A and B the lines parallel to the y- and x-axes respectively will meet at point P of the ellipse. For all ϕ we have $x = d(O,C) = a \cos \phi$, and $y = d(C,P) = d(D,B) = b \sin \phi$.

The equations are

$$\begin{cases} x = a \cos \phi \; , \\ y = b \sin \phi \; . \end{cases}$$

We may eliminate ϕ as follows:

$$\frac{x}{a} = \cos \phi \; , \; \frac{y}{b} = \sin \phi \; ;$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \phi + \sin^2 \phi = 1 \; ,$$

or, $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \; ,$$

which is the usual equation of an ellipse in rectangular coordinates. Note that the parameter ϕ used here is not the angle between the positive part of the x-axis and the radius vector $\overrightarrow{OP}$ to the point P; that is, it is not the angle used in representing P in polar coordinates.

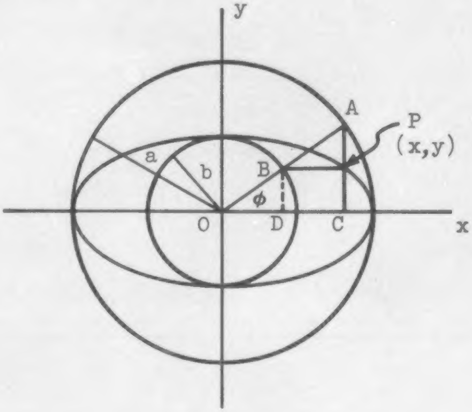


Figure 5-11

It should be recognized that we may select a parameter in various ways to fit a variety of situations. There is never a unique way to do this, so it is inaccurate to refer to "the parametric equations of ...". Rather, we have "a parametric representation of ..." with the understanding that we have made the choices of constants and variables that best suit the hypothesis and our plan of approach to the solution.

Exercises 5-4

1. Write parametric equations for a circle of radius 10 and with center at the origin.

2. Write parametric equations for the path of a point around the circle of Exercise 1. Assume that it starts from the 3 o'clock position and rotates clockwise at the rate of 4 revolutions per second.

3. Write parametric equations for the path of a point at the end of the minute hand of a clock during one hour. Assume the length of the radius to be 6 inches and that the point starts from the 12 o'clock position to which we assign the numbers 0 and 60 . Use minutes as measures of time.

4. Write parametric equations for a circle with center at (4,0) and radius 3 .

5. Write parametric equations for a circle with center at (0,6) and radius 4 .

6. Write parametric equations for the path of a point moving around the circle of Exercise 4. Assume that it starts from its lowest point and moves clockwise at 2 rps .

7. Write parametric equations for the path of a point moving around the circle of Exercise 5. Assume that it starts from its highest point and moves counterclockwise at 3 rps .

Describe in words the motion of a point whose path has the parametric equations given below. Assume t denotes elapsed time in seconds.

8. $$\begin{cases} x = 4 \cos \pi t , \\ y = 4 \sin \pi t . \end{cases}$$

9. $$\begin{cases} x = 6 \cos (\pi t + \frac{\pi}{2}) , \\ y = 6 \sin (\pi t + \frac{\pi}{2} . \end{cases}$$

10. $$\begin{cases} x = 8 \cos (\pi - 3\pi t) , \\ y = 8 \sin (\pi - 3\pi t) . \end{cases}$$

11. $$\begin{cases} x = 10 \cos (\frac{3\pi}{2} + 10\pi t) , \\ y = 10 \sin (\frac{3\pi}{2} + 10\pi t) . \end{cases}$$

12. $\begin{cases} x = 4 + \cos 6\pi t , \\ y = \sin 6\pi t . \end{cases}$

13. $\begin{cases} x = \cos 8\pi t , \\ y = -3 + \sin 8\pi t . \end{cases}$

14. $\begin{cases} x = 2 + \cos 12\pi t , \\ y = 5 + \sin 12\pi t . \end{cases}$

15. $\begin{cases} x = a + b \cos 2\pi t , \\ y = c + b \cos 2\pi t . \end{cases}$

16. $\begin{cases} x = p + q(\cos 2n\pi t - \alpha), \\ y = r + q(\cos 2n\pi t - \alpha). \end{cases}$

17. The equations of motion of a point moving uniformly on a circular path are

$$\begin{cases} x = 6 \cos 4\pi t , \\ y = 6 \sin 4\pi t . \end{cases} \qquad (t \text{ in seconds})$$

(a) Describe its motion in words.

(b) Make a table showing the coordinates of the point at the times $t = 0 , .1 , .2 , \ldots , 1.0$ second.

(c) A second point travels on the same circle in the same direction at the same rate, and starts at the same time, but from the point on the y-axis above the origin. Write equations for its motion.

(d) A third point starts at the same time and place as the first point, but travels in the opposite direction at half its speed. Find equations of motion for this third point.

(e) Find the times and places at which the third point meets the first point, as was done in Examples 2 and 3.

(f) Find the times and places where the third point meets the second point.

18. Three bicyclists, A , B , C are equally spaced around a one mile circular track, (say at the 8 o'clock, 4 o'clock, and 12 o'clock positions, respectively). A and B , who go clockwise, can circle the track in 3 minutes and 4 minutes respectively. C , who travels counterclockwise, can circle the track in 5 minutes. They start at the same moment.

(a) Write equations of motion for their angular positions on the track at any time t after they start.

(b) Find and illustrate their positions at the end of each of the first 10 minutes.

(c) Determine the first 5 meetings; who meet; when, and where?

(d) When and where do all three meet, if ever?

19. A point starts at A (Figure 5-9) and moves counterclockwise at 2 rps. A second point starts at position P, which you are to find, and, moving clockwise at the same rate, passes the first point each time they cross the y-axis. Write the equations of motion for this second point.

20. Four points, P, Q, R, S are equally spaced around a circle (Figure 5-9), with P at the 3 o'clock position, Q at the 12 o'clock position, R at the 9 o'clock position, and S at the 6 o'clock position. P and Q move counterclockwise, R and S clockwise. They start simultaneously, and all meet for the first time 10 seconds later at the 10 o'clock position.

    (a) Write equations of motion for each point.

    (b) When and where will all four meet again?

5-5. Parametric Equations of the Cycloid.

A curve frequently encountered in physical applications is the cycloid. We introduce it in an example.

Example 1. A wheel of radius a feet rolls in a straight line down a flat road. Find analytic conditions for the path of a point P on the rim of the wheel.

Solution. Something--perhaps years of experience--suggests a parametric representation.

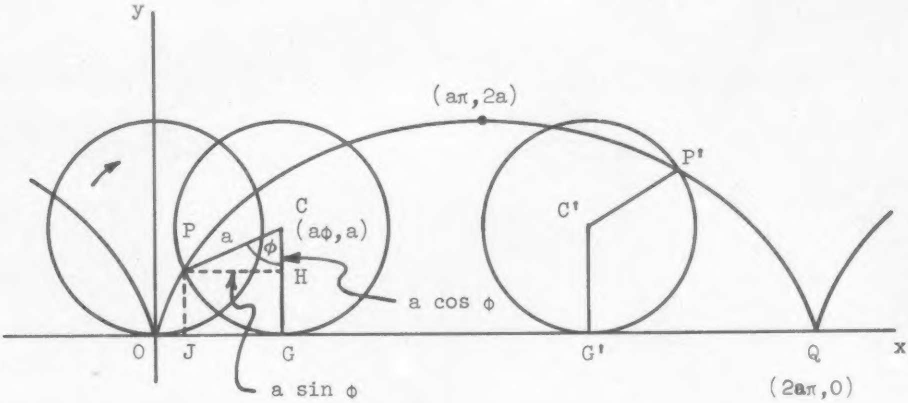


Figure 5-12

Let the line along which the wheel rolls be the x-axis, and let the origin be a point at which P touches the road. Let the positive direction on the x-axis be the direction in which the wheel is rolling. Finally, let ϕ be the radian measure of the angle through which the wheel has rotated since P touched the road, with ϕ positive when the center of the wheel has a positive abscissa. Since the wheel is rolling, not slipping, the length of $\overline{OG}$ is the same as the length of $\widehat{PG}$. The definition of radian measure gives this arc length as $a\phi$. Hence,

$$\begin{cases} x = d(O,J) = d(O,G) - d(P,H) = a\phi - a\sin\phi \,, \\ y = d(P,J) = d(C,G) - d(C,H) = a - a\cos\phi \,. \end{cases}$$

We rewrite these parametric equations of the cycloid

$$(1) \qquad \begin{cases} x = a\phi - a\sin\phi \,, \\ y = a - a\cos\phi \,; \end{cases} \qquad \text{or} \qquad \begin{cases} x = a(\phi - \sin\phi) \,, \\ y = a(1 - \cos\phi) \,. \end{cases}$$

If the wheel were rotating at the rate of ω radians per second, then $\phi = \omega t$ and Equations (1) become

$$(2) \qquad \begin{cases} x = a\omega t - a\sin\omega t \,, \\ y = a - a\cos\omega t \,. \end{cases}$$

Exercises 5-5

1. A point $P = (x,y)$ on the rim of a wheel with a 2 inch diameter traces a cycloid as the wheel rolls along the x-axis. Write parametric equations for the locus of P. Find rectangular coordinates for P, correct to tenths, corresponding to values of θ from 0° to 360° at intervals of 30°. Make a careful drawing of the graph.

2. One arch of a cycloid will just fit inside a rectangle 6 units high. How wide is that rectangle? Choose suitable axes and then write parametric equations for the cycloid.

3. A wheel with a 6 inch diameter is rolling along a line, rotating 4 times per second.

   (a) Choose a suitable coordinate system and write parametric equations of the motion of a point $P = (x,y)$ on the rim.

   (b) Find rectangular coordinates for the positions of P at times $t = .1, .2, .3, .4, .5$.

   (c) Find the time and place at which P first reaches a high point on its path.

4. An automobile traveling along a straight and level road at 30 miles an hour has a wheel whose outer circumference is 66 inches.

   (a) Make an accurate scale drawing of one arch of the cycloid traced by a point on the circumference.

   (b) Choose a suitable coordinate system and write parametric equations for the motion of a point on the rim of the wheel. Use a minute as a unit of time and $3\frac{1}{7}$ as an approximate value for π .

Challenge Exercises for Sections 5-3, 5-4, 5-5

1. (Refer to Figure 5-12.) If, as in the case of a cycloid, we consider a wheel of radius a rolling down a straight flat road, we may consider the path of a point P not on the rim, but along a radius $\overline{CF}$, at a distance of b feet from the center. We distinguish two cases: $b > a$, and $b < a$. The locus in the first case is called a prolate cycloid, and in the second case a curtate cycloid. Figure 5-13 illustrates a case which leads to a prolate cycloid, whose parametric equations you are asked to find. A part of the graph is shown in Figure 5-14.

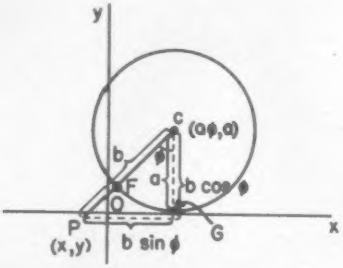
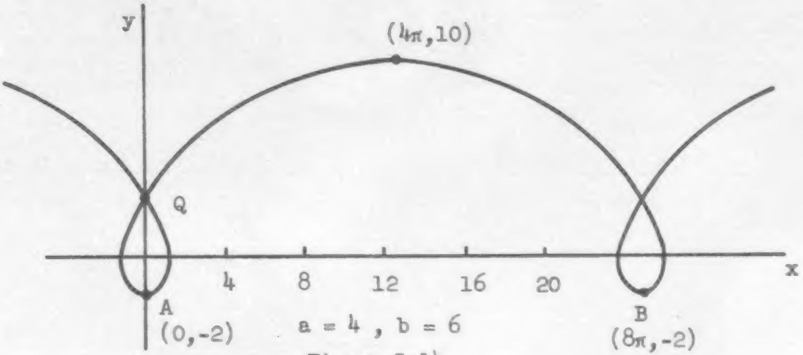


Figure 5-13



$a = 4$, $b = 6$
Figure 5-14

This figure illustrates a case in which $b = 1.5a$. (Can you find the ordinate of the point Q in which the graph cuts the y-axis?) The student is urged to consider the cases: $b = 2a$, $b = 10a$, and to draw some general conclusions.

2. The curtate cycloid. (Refer to Figures 5-13, 5-14.) Find the locus of a point P on the radius $\overline{CF}$ of a circle as the circle rolls along a line. $d(C,P) = b$; radius $= d(C,F) = a$, and $b < a$. Choose a suitable coordinate system and draw an arch of the graph of a curtate cycloid for the case $a = 6$, $b = 4$.

3. A circle of radius a rolls, without slipping, on the outside of a circle of radius b . Find an analytic representation of the locus of a point P on the outside circle.

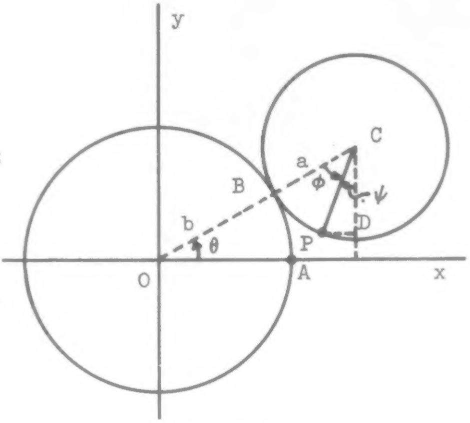


Figure 5-15

Discussion: We illustrate the case $a < b$, and suggest these relations: length of $\widehat{AB}$ = length of $\widehat{PB}$, $\therefore a\phi = b\theta$. $C = \big((a + b)\cos\theta , (a + b)\sin\theta\big)$; the sum of the measures of θ , ϕ , ψ is $\frac{\pi}{2}$ or 90° ; $d(P,D) = a \sin\psi$; $d(C,D) = a\cos\psi$. We urge the student to experiment with the special cases $a = b$, $a = \frac{1}{2}b$, $a = \frac{1}{3}b$. Such curves are called epicycloids and have applications in astronomy and in mechanical engineering.

4. (Refer to the previous problem.) A circle of radius a rolls, without slipping, on the inside of a circle of radius b $(a < b)$. Find analytic representations of the path of a point P on the circumference of the inside circle. Such a path is called a hypocycloid. The student is urged to experiment with the special cases $a = \frac{1}{4}b$, $a = \frac{1}{3}b$, $a = \frac{1}{2}b$. In both this and the previous exercise the student is challenged to answer this question without performing the experiment: If $a = \frac{1}{2}b$, and we make a complete circuit, how many times has the smaller circle rotated on its own axis?

5. A circle of radius a has as center $C = (0,a)$. A chord is drawn through any point $D = (x_1,y_1)$ of the circle and extended to meet, at Q, the tangent to the circle at A, the end of the diameter from O. $\overline{QR}$ is drawn parallel to $\overline{AO}$, and a line is drawn from D parallel to $\overline{AQ}$, and intersecting $\overline{QR}$ at $P = (x,y)$. Find equations of the locus of P as the point D moves on the circle. Sketch the locus. (This curve, called the witch of Agnesi, was studied and named by a mathematician of the eighteenth century, Maria Gaetana Agnesi.)

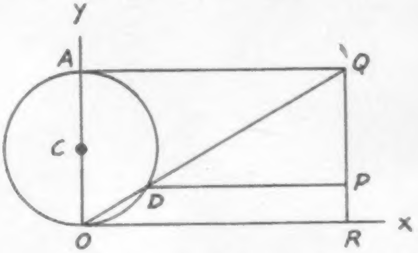


Figure 5-16

6. Find an equation of the locus of a point which moves so that the sum of the squares of its distances from two fixed points is a constant, which we call $2a^2$. Describe and sketch the locus.

7. Find an equation of the locus of a point which moves so that the sum of the squares of its distances from the vertices of a square is constant. Describe the locus.

8. Find an equation of the locus of a point which moves so that the sum of the squares of its distances from the lines containing the sides of a square is constant.

9. A line drawn parallel to the side $\overline{AB}$ of a triangle ABC, meets $\overline{AC}$ in D, $\overline{BC}$ in E. The lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{BD}$ meet at P. Find an equation of the locus consisting of all such points P. (Hint: Let $\overleftrightarrow{AB}$ be the x-axis and let $C = (0,c)$, where $c > 0$. Introduce, as a parameter, t, the distance between $\overline{DE}$ and the x-axis.)

10. Let O and Q be distinct points. Let L be a line through O and let P be the foot of the perpendicular to L through Q . What is the locus of P as L rotates around O ? (Hint: Use the slope of L as an auxiliary variable. Remember that some lines don't have slopes. Does Q lie on the locus?)

11. A circle of radius a has its diameter OCA along the polar axis. From O a chord $\overline{OR}$ is drawn and extended to meet, at S , the tangent to the circle at A . Find equations of the locus of P , a point on $\overrightarrow{OS}$ such that $d(P,S) = d(OR)$. Make a sketch of the graph. (This locus is a cissoid, a curve studied by the Greek mathematician Diocles, who lived a century or so after Euclid. You may learn something more about it when you study inversion later.

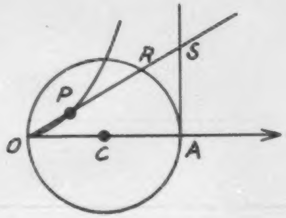


Figure 5-17

12. A fixed line $\overleftrightarrow{BC}$ is perpendicular to the polar axis at point A , a units from the pole. A line is drawn through O meeting $\overleftrightarrow{BC}$ at R . A fixed length ℓ is marked off from R on this line in both directions locating the points P and P' . Find an equation in polar coordinates for the locus of P and P' . (This curve, called a conchoid, was studied by the Greek mathematician Nicomedes about two centuries B.C. It can be used in the trisection of an angle. Try to discover how.)

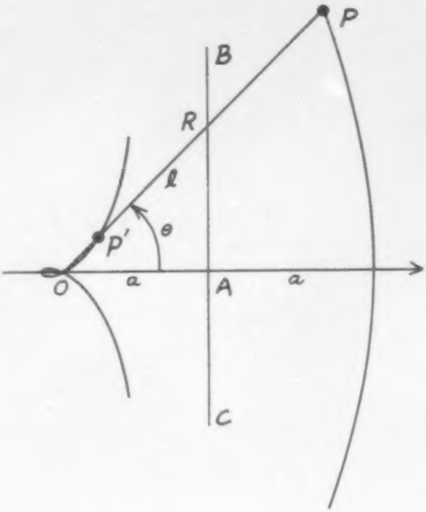


Figure 5-18

13. Involute of the circle. A string of no thickness is wrapped around a fixed circle; the end of the string is at A . We unwrap the string, keeping it taut, and tangent to the circle. ($\overline{PT}$ is tangent to the circle, and $d(P,T)$ = length of $\widehat{AT}$). Find analytic conditions for the graph of P . This graph is called the involute of the circle. Try to generalize this idea, and sketch involutes for an ellipse, a parabola, ... Does every curve have an involute? Make some mechanical models with which you can draw involutes. Draw the involute of a square.

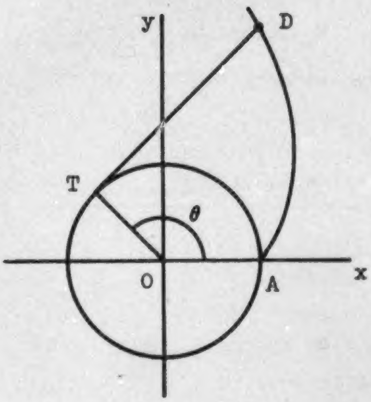


Figure 5-19

14. Suppose a fixed circle with radius a is internally tangent to a circle with radius b $(b > a)$. Find parametric equations for the locus of a point P on the outer circle as the outer circle rolls around the inner circle without slipping.

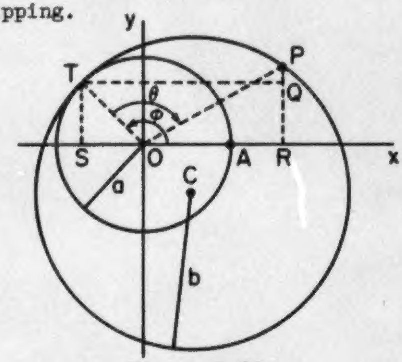


Figure 5-20

5-6. Parametric Equations of a Straight Line.

Parametric representation, which we found so useful in the complicated cases of the previous sections can be used to illuminate and extend the discussion of the straight line. Some of the exercises of Section 5-2 have already introduced you to the ideas and methods we examine now in more detail. The foundations for this discussion have already been developed in Chapter 2, particularly in Section 3, where we find these equations:

$$(1) \qquad \begin{cases} x = x_0 + t(x_1 - x_0) , \\ y = y_0 + t(y_1 - y_0) . \end{cases}$$

We recognize that the quantities $x_1 - x_0$ and $y_1 - y_0$ are direction numbers obtained from the coordinates of the point $P_0 = (x_0,y_0)$ and $P_1 = (x_1,y_1)$. Therefore, we represent them respectively by ℓ and m, and rewrite Equations (1) as

$$\left\{\begin{aligned} x &= x_0 + \ell t , \\ y &= y_0 + mt . \end{aligned}\right. \tag{2}$$

We recognize that t is a parameter, and that these equations are parametric equations of the line through the points P_0 and P_1, which we assume to be distinct.

If $x_1 = x_0$, then $y_1 \neq y_0$, and (2) takes the form

$$\left\{\begin{aligned} x &= x_0 , \\ y &= y_0 + mt . \end{aligned}\right.$$

(What is the geometric version of this hypothesis and conclusion?)

If $y_1 = y_0$, then $x_1 \neq x_0$, and (2) takes the form

$$\left\{\begin{aligned} x &= x_0 + \ell t , \\ y &= y_0 . \end{aligned}\right.$$

(What is the geometric version of this hypothesis and conclusion?)

Example 1. Find a parametric representation of the line through (2,0) and (-4,3).

Solution: We can choose either point as P_0. If $P_0 = (2,0)$ then $x_1 - x_0 = -6$, $y_1 - y_0 = 3$ and we get the representation

$$\left\{\begin{aligned} x &= 2 - 6t , \\ y &= 0 + 3t . \end{aligned}\right.$$

The other choice for P_0 leads to the representation

$$\left\{\begin{aligned} x &= -4 + 6t , \\ y &= 3 - 3t . \end{aligned}\right.$$

A parametric representation of a line sets up a one-to-one correspondence between the real numbers and the points on a line in the plane. We illustrate below the correspondences established by the parametric representations we found for the line in Example 1.

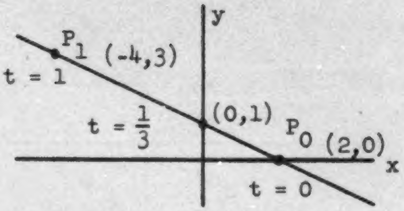


Figure 5-21a

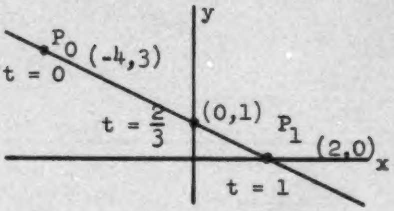


Figure 5-21b

Example 2. Find the intersection of the line through $(4,2)$ and $(2,-4)$ and the line through $(-3,-1)$ and $(-4,2)$.

Solution: The lines may be represented parametrically as follows:

$$L_1: \begin{cases} x = 4 - 2s , \\ y = 2 - 6s . \end{cases} \qquad L_2: \begin{cases} x = -3 - t , \\ y = -1 + 3t . \end{cases}$$

We wish to find all points which lie on both lines. Now the point (x,y) lies on both lines if and only if there exist values s_0 and t_0 of s and t such that

$$x = 4 - 2s_0 = -3 - t_0 ,$$

$$y = 2 - 6s_0 = -1 + 3t_0 .$$

All such values of s and t can be found by solving simultaneously the equations

$$4 - 2s = -3 - t ,$$

$$2 - 6s = -1 + 3t .$$

The only solutions are $s = 2$, $t = -3$. Substituting these in either pair of parametric equations, we find that the only point of intersection is $(0,-10)$.

It would have been quite correct to use the same letter for the parameter in the parametric representations of L_1 and L_2 . However, this would have led to difficulties later in the problem. Do you see why? Can you find another method of getting around the difficulties?

In previous sections of this chapter we related the parameter t to elapsed time. In such cases the parametric equations gave us equations of motion of the point P. The graph of these equations was directly related to the path of the point. Example 3 shows how this approach can be used for the line.

Example 3. A ball is rolling along a level surface in a straight line with constant velocity. The surface is provided with a Cartesian coordinate system with the foot as the unit of length. At 10:00 a.m. the ball is at $(4,2)$ while one second later it is at $(2,-4)$. A second ball, also rolling along the level surface in a straight line with a constant velocity, is at $(-4,2)$ at 10:00 a.m., at $(-3,-1)$ one second later. We ask whether the two balls will collide. In other words, we want to know not whether their paths intersect but whether, if they do, the two balls are at any point of intersection at the same time. We assume, in order to simplify the problem, that the balls have zero radii and will collide only if their centers coincide.

Solution. The path of the first ball is represented by the equations

$$\begin{cases} x = 4 - 2s \,, \\ y = 2 - 6s \,. \end{cases}$$

Further, if s is the number of seconds which have elapsed since 10:00 a.m., the equations also tell us where the ball is at any time. For if we set $s = 0$ (10:00:00 a.m.) we get $x = 4$ and $y = 2$, while if we set $s = 1$ (10:00:01 a.m.) we get $x = 2$ and $y = -4$. Further, in s seconds starting at 10:00:00 a.m., an object whose motion was represented by these equations would travel

$$\sqrt{(x - 4)^2 + (y - 2)^2} = \sqrt{4 + 36}\, s = 2\sqrt{10}\, s$$

feet. Thus the distance travelled is a constant multiple of the time taken and the speed is constant. Similarly, the motion of the second ball is described by the equations

$$\begin{cases} x = -4 + t \,, \\ y = 2 - 3t \,. \end{cases}$$

Our problem is to find out whether the abscissas of the positions of the two balls, and the ordinates, are ever simultaneously $(s = t)$ equal. In other words we ask whether the system of equations

$$\begin{cases} 4 - 2t = -4 + t , \\ 2 - 6t = 2 - 3t . \end{cases}$$

has a solution. Clearly not, since this pair is equivalent to the pair

$$\begin{cases} 3t = 8 , \\ 3t = 0 . \end{cases}$$

Thus the balls do not collide.

If direction cosines are used in a parametric representation of a line, the parameter t has an interesting interpretation. Since

$$d(P_0,P) = \sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{\lambda^2 t^2 + \mu^2 t^2} = |t| ,$$

the absolute value of the parameter is the distance of the corresponding point P from P_0 .

Example 4. Find, on the line through $P_0 = (1,5)$ and $P_1 = (5,8)$, two points which are 3 units distant from P_0 .

Solution. Direction numbers for $\overleftrightarrow{P_0P_1}$ are $(4,3)$, and direction cosines can be taken as $\left(\frac{4}{5} , \frac{3}{5}\right)$. We may then write parametric equations for $\overleftrightarrow{P_0P_1}$ in terms of direction cosines as

$$\begin{cases} x = 1 + \frac{4}{5}t , \\ y = 5 + \frac{3}{5}t . \end{cases}$$

The substitution $t = \pm 3$ gives the coordinates of both points, $\left(1 \pm \frac{12}{5} , 5 \pm \frac{9}{5}\right)$, or $(3.4,6.8)$ and $(-1.4,3.2)$

Exercises 5-6

1. Find two parametric representations for each line through one of the following pairs of points, using each pair in both possible orders.

   (a) $(5,-1)$, $(2,3)$
   (b) $(0,0)$, $(4,1)$
   (c) $(2,-3)$, $(2,3)$
   (d) $(-1,4)$, $(-6,4)$
   (e) $(1,1)$, $(2,2)$
   (f) $(-1,-1)$, $(1,1)$
   (g) $(1,0)$, $(0,1)$
   (h) $(2,-2)$, $(-2,2)$

2. Draw the graph of each of the lines in Exercise 1, plotting, on each, the points corresponding to the values -1 , 0 , 1 , and 2 of the parameter.

3. Find the intersection of each of the following pairs of lines. When the lines do not intersect, what do you notice about their equations?

   (a) $\begin{cases} x = 5 + s \\ y = 2 - s \end{cases}$ $\qquad$ $\begin{cases} x = 4 - 2t \\ y = -6 + 3t \end{cases}$

   (b) $\begin{cases} x = 2 - 3s \\ y = 1 + 2s \end{cases}$ $\qquad$ $\begin{cases} x = 4 + 6t \\ y = -5 - 4t \end{cases}$

   (c) $\begin{cases} x = -3 + s \\ y = 2 - 3s \end{cases}$ $\qquad$ $\begin{cases} x = -2 - t \\ y = -1 + 3t \end{cases}$

4. Find a pair of parametric equations for the line L with equation $2x - 3y + 1 = 0$.

5. Let L have the parametric equations
$$\begin{cases} x = x_0 + \ell t \ , \\ y = y_0 + mt \ . \end{cases}$$
   Let $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ be the points on L given by $t = t_1$, and $t = t_2$, respectively. Prove that $d(P_1,P_2) = \sqrt{\ell^2 + m^2}\,|t_2 - t_1|$.

6. A ball is rolling on a level floor along the line through $(16,2)$ and $(4,7)$ and in the direction from the first point towards the second. (The unit of length is the foot.) Its speed is 26 feet per second. Find parametric equations for its motion, measuring time from the instant when it is at $(16,2)$.

7. Let S be a set of points in a plane. A point P is sometimes called a center of S if S is symmetric about P. A parametric representation of a line may be used to prove that a point is a center of a set of points. Let S be the circle with equation $x^2 + y^2 = 4$. Any line through the origin has a parametric representation $x = \lambda t$, $y = \mu t$, with $\lambda^2 + \mu^2 = 1$. Substituting these expressions for x and y in the equation of the circle we get

$$\lambda^2 t^2 + \mu^2 t^2 = 4 ,$$

or
$$t^2 = 4 .$$

Thus
$$t = \pm 2 .$$

Since the answer is independent of λ and μ, every line through the origin meets the circle in the points given by $t = -2$ and $t = 2$. These are equidistant from the origin.

(a) Show that the origin is a center for $b^2x^2 + a^2y^2 = a^2b^2$.

(b) Show that the origin is a center for $y = ax^3$. (Discuss the case when $a > 0$ and the case when $a < 0$.)

(c) Show that the origin is a center for $y = \dfrac{x^3}{x^2 - 1}$.

8. A set S of points in a plane is called bounded if there is a rectangle which contains S. Prove that a bounded set in a plane has at most one center. Is this also true for unbounded sets?

9. Find, on the line through $P_0 = (1,5)$ and $P_1 = (5,8)$, two points at unit distance from P_1.

10. Find, on the line through $A = (-3,5)$ and $B = (0,9)$, two points P and Q such that $d(B,P) = d(B,Q) = 5d(A,B)$.

5-7. Summary.

We have investigated the relations between certain geometric and algebraic entities. The geometric objects were sets of points not, as we have said, given to us in a basket but determined by certain conditions or descriptions. The corresponding algebraic expressions were statements of equality or inequality. The relations between them were approached through a coordinatization of the "space" in which the sets were presented to us. Then our knowledge and ingenuity and experience led us to an algebraic description of the set, in the terminology of our coordinate system.

We have shown this process in detail in a number of situations. We have applied parametric representation in situations involving angular displacement and motions along a circle or line. If a set of points has any special properties or geometric appearance, how is this reflected in its analytic representation? If, for example, the set of points is symmetric in any way, could we tell that from its equation? If, on the other hand, some analytic representation shows a particular algebraic property, what is the geometric counterpart? What would be the geometric effect of imposing certain restrictions on the domain or range of the variables that appear in the analytic representations?

In our next chapter we will investigate in detail many such relations between curves and their analytic representations.

Review Exercises

1. We describe certain sets of points. You are asked to give an analytic description of each.

   (a) All points equidistant from the x- and y-axes.
   (b) All points equidistant from the points $A = (5,0)$ and $B = (11,0)$.
   (c) All points equidistant from $A = (5,0)$ and $C = (5,8)$.
   (d) All points equidistant from $C = (5,8)$ and $B = (11,0)$.
   (e) All points at distance 3 from $C = (5,8)$.
   (f) All points at distance 3 from the line $x = 5$.
   (g) All points at distance 3 from the line $y = -2$.
   (h) All points at distance 3 from the line $3x - 4y + 7 = 0$.
   (i) All points at distance h from the line $x = k$.
   (j) All points at distance p from the line $y = q$.
   (k) All points at distance d from the line $ax + by + c = 0$.
   (l) All points twice as far from $A = (5,0)$ as from $B = (11,0)$.
   (m) All points equidistant from the point $C = (5,8)$ and the x-axis.
   (n) All points equidistant from the point $A = (5,0)$ and the line $x = 1$.
   (o) All points equidistant from the point $D = (5,3)$ and the line $3x - 4y + 7 = 0$.
   (p) All points equidistant from the line $ax + by + c = 0$ and the point $P = (r,s)$ not on that line.

200

2. If $A = (-3,1)$, $B = (5,3)$, $C = (1,5)$, find an analytic representation of

   (a) $\overleftrightarrow{AB}$

   (b) $\overrightarrow{AB}$

   (c) $\overline{AB}$

   (d) $\overleftrightarrow{BC}$

   (e) $\overrightarrow{BC}$

   (f) $\overline{BC}$

   (g) $\overleftrightarrow{CA}$

   (h) $\overrightarrow{CA}$

   (i) $\overline{CA}$

   (j) the interior of $\angle ABC$.

   (k) the interior of $\angle BCA$.

   (l) the interior of $\angle CAB$.

   (m) the interior of $\triangle ABC$.

   (n) the line through A and parallel to $\overleftrightarrow{BC}$.

   (o) the line through B and parallel to $\overleftrightarrow{CA}$.

   (p) the line through C and parallel to $\overleftrightarrow{AB}$.

   (q) the line containing altitude $\overline{AD}$ of $\triangle ABC$.

   (r) the line containing altitude $\overline{BE}$ of $\triangle ABC$.

   (s) the line containing altitude $\overline{CF}$ of $\triangle ABC$.

   (t) the line containing the median of $\triangle ABC$ through A .

   (u) the line containing the median of $\triangle ABC$ through B .

   (v) the line containing the median of $\triangle ABC$ through C .

   (w) the pair of lines through A and parallel to the axes.

   (x) the perpendicular bisector of $\overline{AB}$.

   (y) the perpendicular bisector of $\overline{BC}$.

   (z) the circle containing A , B , and C .

3. The following expressions are analytic descriptions of certain sets. You are asked to describe each set in words, giving its name, its location on the plane, and any special geometric properties it may have. Sketch the graph of each.

   (a) $\frac{x}{3} + \frac{y}{5} = 1$

   (b) $\frac{x}{3} + \frac{y}{5} = 5$

   (c) $x^2 = 16$

   (d) $x^2 + y^2 = 16$

   (e) $x^2 + 9y^2 = 16$

   (f) $x^2 - 9y^2 = 16$

   (g) $x^2 - 9y = 16$

   (h) $9y - x^2 = 16$

   (i) $y^2 - 9x = 16$

   (j) $|x - 3| = 5$

   (k) $|x + 5| < 4$

   (l) $|x - a| \leq b$

   (m) $xy = 0$

   (n) $(x - 1)(y + 2) = 0$

   (o) $x^2 - 3x - 10 = 0$

   (p) $x < y$

   (q) $x^2 < y^2$

   (r) $x < x^2$

4. Give verbal descriptions of each of the sets described analytically with polar coordinates below. Give its name if available, its location on the plane, and any special geometric properties it may have.

(a) $r^2 = 9$

(b) $r^2 < 9$

(c) $r < 3$

(d) $r > 3$

(e) $\theta = 2$

(f) $\theta < \frac{\pi}{2}$

(g) $r = 2\theta$

(h) $r < \theta$

(i) $|\theta - 2| = .1$

(j) $|r - 5| < .1$

(k) $r = \frac{6}{\sin \theta}$

(l) $r = \frac{-3}{\cos \theta}$

(m) $r = \frac{-2}{\cos \theta}$

(n) $r = \frac{5}{\cos \theta}$

(o) $r = \frac{1}{\cos(\theta + \frac{\pi}{4})}$

(p) $r = \frac{4}{\sin(\theta - \frac{\pi}{4})}$

(q) $r = \frac{a}{\sin(\theta - b)}$

(r) $r > \frac{1}{\sin \theta}$

(s) $r < \frac{2}{\cos \theta}$

(t) $r = 0$.

5. Write the related polar equation or inequality for each part of Exercise 4 above.

6. Eliminate the parameter in each pair of parametric equations below.

(a) $\begin{cases} x = 1 + t \\ y = 1 + t^2 \end{cases}$

(b) $\begin{cases} x = 2t \\ y = t + 2 \end{cases}$

(c) $\begin{cases} x = \frac{1}{t + 1} \\ y = \frac{1}{2t + 1} \end{cases}$

(d) $\begin{cases} x = t^2 + t \\ y = t^3 + t^2 \end{cases}$

(e) $\begin{cases} x = t + \frac{1}{t} \\ y = t^2 + \frac{1}{t^2} \end{cases}$

(f) $\begin{cases} x = 3 \sin t \\ y = r \cos t \end{cases}$

(g) $\begin{cases} x = 2 + 3 \cos t \\ y = 4 - 5 \sin t \end{cases}$

(h) $\begin{cases} x = 2 \sin t \\ y = \sin 2t \end{cases}$

(i) $\begin{cases} x = \frac{1}{\sin t} \\ y = \frac{1}{\cos t} \end{cases}$

(j) $\begin{cases} x = \sin 2t \\ y = \sin \frac{1}{2}t \end{cases}$

7. A point moves on a line from $A = (3,7)$ through $B = (0,3)$ at the rate of 1 linear unit per second. Write parametric equations for its path, using seconds as units for the parameter t.

8. A point moves on a line from the origin through point $C = (7,24)$ at the rate of 5 linear units per second. Write parametric equations for its path, using minutes as units for the parameter t.

9. A point A moves along a line with parametric equations for its path: $\begin{cases} x = -1 + 3t, \\ y = 3 - t. \end{cases}$ Point B moves along a line with parametric equations for its path: $\begin{cases} x = 5 - 2t, \\ y = 11 + t. \end{cases}$ Find $d(A,B)$ when $t = 3$, and when $t = 5$.

10. The path of P_1 has equations $\begin{cases} x = x_1 + \ell_1 t, \\ y = y_1 + m_1 t. \end{cases}$

The path of P_2 has equations $\begin{cases} x = x_2 + \ell_2 t, \\ y = y_2 + m_2 t. \end{cases}$

Express $d(P_1,P_2)$ when $t = 2$, in terms of the constants in these equations.

11. Write parametric equations for each path of a point around the rim of a clock if the path has the following description (assume unit radius):

(a) Starts at 12 o'clock position, and moves counterclockwise at 3 rps (revolutions per second).
(b) Starts at 6 o'clock position and moves clockwise at 2 rps.
(c) Starts at 4 o'clock position and moves counterclockwise at 1 rps.
(d) Starts at 9 o'clock position and moves clockwise at 4 rps.
(e) Starts at 8 o'clock position and moves counterclockwise at $\frac{1}{2}$ rps.

12. Find the time and place of the first meeting, assuming a simultaneous start of the points described in Exercise 11:

(a) a and b
(b) a and c
(c) a and d
(d) a and e
(e) b and c
(f) b and d
(g) b and e
(h) c and d
(i) c and e
(j) d and e

13. A point is rotating counterclockwise at 2 rps at a distance 3 from the point $(4,5)$. Find analytic conditions for its path.

14. A point is rotating clockwise at 1 rps at a distance of 2 from the point $(-1,0)$. Find analytic conditions for its path.

15. We give analytic descriptions of the paths of certain points around the rim of a clock. You are asked to describe these parts in words. Assume t measured in minutes.

(a) $$\begin{cases} x = 4 \cos 4\pi t \\ y = 4 \sin 4\pi t \end{cases}$$

(b) $$\begin{cases} x = 6 \cos(\frac{\pi}{2} + 6\pi t) \\ y = 6 \sin(\frac{\pi}{2} + 6\pi t) \end{cases}$$

(c) $$\begin{cases} x = 10 \cos(\pi - 10\pi t) \\ y = 10 \sin(\pi - 10\pi t) \end{cases}$$

(d) $$\begin{cases} x = 8 \cos(4\pi t + \pi) \\ y = 8 \sin(4\pi t + \pi) \end{cases}$$

(e) $$\begin{cases} x = 2 \sin 2\pi t \\ y = 2 \cos 2\pi t \end{cases}$$

16. Find parametric representations for the ellipses described below:

(a) center at the origin, major axis 10 along the x-axis, minor axis 6.

(b) center at the origin, x-intercepts ± 3, y-intercepts ± 4.

(c) major axis horizontal, and the ellipse will just fit between the circles $x^2 + y^2 = 5$ and $x^2 + y^2 = 6$.

17. A wheel with radius 12 inches, turning at the rate of 3 rps, is rolling down a straight, level road. Assume a coordinate system as usual and write parametric equations for

(a) a point P on its rim;

(b) a point Q, six inches in from the rim. (A challenge problem.)

Table I

Natural Trigonometric Functions (Degree Measure)

Deg.	Sine	Cosine	Tangent	Cotangent	
0	0.000	1.000	0.000	*****	90
1	0.017	1.000	0.017	57.29	89
2	0.035	0.999	0.035	28.64	88
3	0.052	0.999	0.052	19.08	87
4	0.070	0.998	0.070	14.30	86
5	0.087	0.996	0.087	11.43	85
6	0.105	0.995	0.105	9.514	84
7	0.122	0.993	0.123	8.144	83
8	0.139	0.990	0.141	7.115	82
9	0.156	0.988	0.158	6.314	81
10	0.174	0.985	0.176	5.671	80
11	0.191	0.982	0.194	5.145	79
12	0.208	0.978	0.213	4.705	78
13	0.225	0.974	0.231	4.331	77
14	0.242	0.970	0.249	4.011	76
15	0.259	0.966	0.268	3.732	75
16	0.276	0.961	0.287	3.487	74
17	0.292	0.956	0.306	3.271	73
18	0.309	0.951	0.325	3.078	72
19	0.326	0.946	0.344	2.904	71
20	0.342	0.940	0.364	2.747	70
21	0.358	0.934	0.384	2.605	69
22	0.375	0.927	0.404	2.475	68
23	0.391	0.921	0.424	2.356	67
24	0.407	0.914	0.445	2.246	66
25	0.423	0.906	0.466	2.145	65
26	0.438	0.899	0.488	2.050	64
27	0.454	0.891	0.510	1.963	63
28	0.469	0.883	0.532	1.881	62
29	0.485	0.875	0.554	1.804	61
30	0.500	0.866	0.577	1.732	60
31	0.515	0.857	0.601	1.664	59
32	0.530	0.848	0.625	1.600	58
33	0.545	0.839	0.649	1.540	57
34	0.559	0.829	0.675	1.483	56
35	0.574	0.819	0.700	1.428	55
36	0.588	0.809	0.727	1.376	54
37	0.602	0.799	0.754	1.327	53
38	0.616	0.788	0.781	1.280	52
39	0.629	0.777	0.810	1.235	51
40	0.643	0.766	0.839	1.192	50
41	0.656	0.755	0.869	1.150	49
42	0.669	0.743	0.900	1.111	48
43	0.682	0.731	0.933	1.072	47
44	0.695	0.719	0.966	1.036	46
45	0.707	0.707	1.000	1.000	45
	Cosine	Sine	Cotangent	Tangent	Deg.

Table II

Natural Trigonometric Functions (Radian Measure)

Rad.	Sine	Cosine	Tangent	Cotangent
.00	0.000	1.000	0.000	*****
.02	0.020	1.000	0.020	49.99
.04	0.040	0.999	0.040	24.99
.06	0.060	0.998	0.060	16.65
.08	0.080	0.997	0.080	12.47
.10	0.100	0.995	0.100	9.967
.12	0.120	0.993	0.121	8.293
.14	0.140	0.990	0.141	7.096
.16	0.159	0.987	0.161	6.197
.18	0.179	0.984	0.182	5.495
.20	0.199	0.980	0.203	4.933
.22	0.218	0.976	0.224	4.472
.24	0.238	0.971	0.245	4.086
.26	0.257	0.966	0.266	3.759
.28	0.276	0.961	0.288	3.478
.30	0.296	0.955	0.309	3.233
.32	0.315	0.949	0.331	3.018
.34	0.333	0.943	0.354	2.827
.36	0.352	0.936	0.376	2.657
.38	0.371	0.929	0.399	2.504
.40	0.389	0.921	0.423	2.365
.42	0.408	0.913	0.447	2.239
.44	0.426	0.905	0.471	2.124
.46	0.444	0.896	0.495	2.018
.48	0.462	0.887	0.521	1.921
.50	0.479	0.878	0.546	1.830
.52	0.497	0.868	0.573	1.747
.54	0.514	0.858	0.599	1.668
.56	0.531	0.847	0.627	1.595
.58	0.548	0.836	0.655	1.526
.60	0.565	0.825	0.684	1.462
.62	0.581	0.814	0.714	1.401
.64	0.597	0.802	0.745	1.343
.66	0.613	0.790	0.776	1.289
.68	0.629	0.778	0.809	1.237
.70	0.644	0.765	0.842	1.187
.72	0.659	0.752	0.877	1.140
.74	0.674	0.738	0.913	1.095
.76	0.689	0.725	0.950	1.052
.78	0.703	0.711	0.989	1.011
.80	0.717	0.697	1.030	0.971
.82	0.731	0.682	1.072	0.933
.84	0.745	0.667	1.116	0.896
.86	0.758	0.652	1.162	0.861
.88	0.771	0.637	1.210	0.827
.90	0.783	0.622	1.260	0.794

Table II

Natural Trigonometric Functions (Radian Measure)

Rad.	Sine	Cosine	Tangent	Cotangent
.92	0.796	0.606	1.313	0.761
.94	0.808	0.590	1.369	0.730
.96	0.819	0.574	1.428	0.700
.98	0.830	0.557	1.491	0.671
1.00	0.841	0.540	1.557	0.642
1.02	0.852	0.523	1.628	0.614
1.04	0.862	0.506	1.704	0.587
1.06	0.872	0.489	1.784	0.560
1.08	0.882	0.471	1.871	0.534
1.10	0.891	0.454	1.965	0.509
1.12	0.900	0.436	2.066	0.484
1.14	0.909	0.418	2.176	0.460
1.16	0.917	0.399	2.296	0.436
1.18	0.925	0.381	2.427	0.412
1.20	0.932	0.362	2.572	0.389
1.22	0.939	0.344	2.733	0.366
1.24	0.946	0.325	2.912	0.343
1.26	0.952	0.306	3.113	0.321
1.28	0.958	0.287	3.341	0.299
1.30	0.964	0.268	3.602	0.278
1.32	0.969	0.248	3.903	0.256
1.34	0.973	0.229	4.256	0.235
1.36	0.978	0.209	4.673	0.214
1.38	0.982	0.190	5.177	0.193
1.40	0.985	0.170	5.798	0.172
1.42	0.989	0.150	6.581	0.152
1.44	0.991	0.130	7.602	0.132
1.46	0.994	0.111	8.989	0.111
1.48	0.996	0.091	10.98	0.091
1.50	0.997	0.071	14.10	0.071
1.52	0.999	0.051	19.67	0.051
1.54	1.000	0.031	32.46	0.031
1.56	1.000	0.011	92.62	0.011
1.58	1.000	-0.009	-108.65	-0.009
1.60	1.000	-0.029	-34.23	-0.029
1.62	0.999	-0.049	-20.31	-0.049
1.64	0.998	-0.069	-14.43	-0.069
1.66	0.996	-0.089	-11.18	-0.089
1.68	0.994	-0.109	-9.121	-0.110
1.70	0.992	-0.129	-7.697	-0.130
1.72	0.989	-0.149	-6.652	-0.150
1.74	0.986	-0.168	-5.853	-0.171
1.76	0.982	-0.188	-5.222	-0.191
1.78	0.978	-0.208	-4.710	-0.212
1.80	0.974	-0.227	-4.286	-0.233